Supplementary Appendix: Policy Evaluation during a Pandemic

Brantly Callaway[*] Tong Li[†]

May 14, 2021

This supplementary appendix contains additional theoretical results related to estimation and inference of the parameters of interest in the main text.

SA Additional Theoretical Results

This section contains several additional results related to verifying double robustness and inference. The results in this section are, for the most part, not new, but rather slightly adapt existing arguments to the particular case considered in the paper. We provide these results primarily to complete the arguments presented in the paper.

We consider the case where a researcher implements parametric working models for the propensity score and outcome regression. We denote the propensity score working model by $p(\mathcal{F}_{t^*-1}; \pi)$ where π is a finite dimensional parameter, and we denote the pseudo true value of the parameter by π^* and the estimated value of the parameter by $\hat{\pi}$. Likewise, we denote the outcome regression working model by $m^C_{0,t}(\mathcal{F}_{t^*-1}; \mu_t)$ where μ_t is a finite dimensional parameter, and we denote the pseudo true value of the parameter by μ^*_t and the estimated value of the parameter by $\hat{\mu}_t$.

We make the following assumptions

Assumption SA.1 (Random Sample)**.** *The data consists of* $\{Y_{l1}, Y_{l2}, \ldots, Y_{l\mathcal{T}}, \mathcal{F}_{l1}, \mathcal{F}_{l2}, \ldots, \mathcal{F}_{l\mathcal{T}}, D_l\}_{l=1}^n$ *which are iid across locations.*

Assumption SA.2 (Assumptions for propensity score)**.** *(i)* $p(\mathcal{F}_{t^*-1}; \pi) = \Lambda(h_{ps}(\mathcal{F}_{t^*-1})'\pi)$ *is a parametric working model for* $p(\mathcal{F}_{t^*-1})$ *where* $\Lambda(z) = 1/(1+\exp(z))$ *and* h_{ps} *allows for transformations of* $\mathcal{F}_{t^*-1}$*, (ii) the pseudo true value* π^* *is in the interior of the parameter space* Π *which is a compact subset of* $\mathbb{R}^k$ *(k being the dimension of* $h_{ps}(\mathcal{F}_{t^*-1})$*), (iii)* $Q_{ps} := \mathbb{E}[h_{ps}(\mathcal{F}_{t^*-1})h_{ps}(\mathcal{F}_{t^*-1})'\Lambda(h_{ps}(\mathcal{F}_{t^*-1})'\pi^*)(1-\Lambda(h_{ps}(\mathcal{F}_{t^*-1})'\pi^*)]$ *is positive definite, and (iv)* $\mathbb{E}\left[||h_{ps}(\mathcal{F}_{t^*-1})||^4\right] < \infty$*.*

Assumption SA.3 (Assumptions for outcome regression)**.** *For all* $t = t^*, \ldots, \mathcal{T}$*, (i)* $m_{0,t}(\mathcal{F}_{t^*-1}; \mu_t) = h_{or}(\mathcal{F}_{t^*-1})'\mu_t$ *is a parametric working model for* $m^C_{0,t}(\mathcal{F}_{t^*-1})$ *and* h_{or} *allows for transformations of* $\mathcal{F}_{t^*-1}$*, (ii)* $Q_{0,or} := \mathbb{E}[h_{or}(\mathcal{F}_{t^*-1})h_{or}(\mathcal{F}_{t^*-1})'|D=0]$ *is positive definite, (iii)* $\mathbb{E}[C_t^4|D=0] < \infty$*, (iv)* $\mathbb{E}\left[||h_{or}(\mathcal{F}_{t^*-1})||^4|D=0\right] < \infty$

Assumption SA.4. *For all* $t = t^*, \ldots, \mathcal{T}$*, at least one (but not necessarily both) of the following conditions hold: (i)* $p(\mathcal{F}_{t^*-1}) = p(\mathcal{F}_{t^*-1}; \pi^*)$*, (ii)* $m^C_{0,t}(\mathcal{F}_{t^*-1}) = m^C_{0,t}(\mathcal{F}_{t^*-1}; \mu^*_t)$

[*]Department of Economics. University of Georgia. brantly.callaway@uga.edu
[†]Department of Economics. Vanderbilt University. tong.li@vanderbilt.edu

Assumptions SA.1 to SA.4 are all standard assumptions. Assumption SA.2 says that we use a logit model for the propensity score working model and invokes standard conditions for logit models. Assumption SA.3 says that we use a linear model for the outcome regression working model and invokes standard assumptions for linear models. Assumption SA.4 says that at least one of the propensity score working model or outcome regression working model is correctly specified.

It is helpful to define some additional notation regarding the weights in Theorem 2. First,

$$\omega_1(D) := \frac{D}{\mathbb{E}[D]} \qquad \text{and} \qquad \hat{\omega}_1(D) := \frac{D}{\bar{D}}$$

where $\bar{D} = n^{-1}\sum_{l=1}^{n} D_l$. In addition, define

$$\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}, \pi) := \frac{p(\mathcal{F}_{t^*-1}; \pi)}{(1 - p(\mathcal{F}_{t^*-1}; \pi))}(1 - D)$$

$$\omega_0(D, \mathcal{F}_{t^*-1}, \pi) := \frac{\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}, \pi)}{\mathbb{E}[\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}, \pi)]}$$

$$\hat{\omega}_0(D, \mathcal{F}_{t^*-1}; \pi) := \frac{\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}; \pi)}{\frac{1}{n}\sum_{h=1}^{n} \tilde{\omega}_0(D_h, \mathcal{F}_{ht^*-1}; \pi)}$$

We also denote the estimated weights by $\hat{\omega}(D, \mathcal{F}_{t^*-1}, \hat{\pi}) = \hat{\omega}_1(D) - \hat{\omega}_0(D, \mathcal{F}_{t^*-1}, \hat{\pi})$.

The next result shows that the estimand in Equation (9) in Theorem 2 is indeed doubly robust.

Proposition SA.1. *Under the Stochastic SIRD Model for Untreated Potential Outcomes and Assumptions 1 and SA.1 to SA.4,*

$$\widehat{ATT}_t^C = \frac{1}{n}\sum_{l=1}^{n} \hat{\omega}(D_l, \mathcal{F}_{lt^*-1}, \hat{\pi})(C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \hat{\mu}_t) \xrightarrow{p} ATT_t^C$$

Proof. Under Assumptions SA.2 and SA.3, it immediately holds by the weak law of large numbers and continuous mapping theorem that

$$\widehat{ATT}_t^C \xrightarrow{p} \mathbb{E}\left[\omega(D, \mathcal{F}_{t^*-1}, \pi^*)(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*))\right]$$

It remains to show that this expression is equal to ATT_t^C.

Case 1: Propensity Score is Correctly Specified

First, consider the case where the propensity score is correctly specified so that $p(\mathcal{F}_{t^*-1}) = p(\mathcal{F}_{t^*-1}; \pi^*)$, but where the outcome regression may be misspecified so that it can be the case that $m_{0,t}^C(\mathcal{F}_{t^*-1}) \neq m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*)$. In this case, $\omega(D, \mathcal{F}_{t^*-1}) = \omega(D, \mathcal{F}_{t^*-1}; \pi^*)$. From the arguments in Equation (20) in the proof of Theorem 2, it holds that

$$ATT_t^C = \mathbb{E}[\omega(D, \mathcal{F}_{t^*-1})C_t] \tag{SA.1}$$

Then, notice that

$$\begin{aligned}\mathbb{E}[\omega(D, \mathcal{F}_{t^*-1})m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*)] &= \mathbb{E}\left[m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*)\mathbb{E}[\omega(D, \mathcal{F}_{t^*-1})|\mathcal{F}_{t^*-1}]\right] \\ &= 0 \end{aligned} \tag{SA.2}$$

where the last equality holds by Equation (21).

Combining, Equations (SA.1) and (SA.2) implies the result for this part when the propensity score is correctly specified.

Case 2: Outcome Regression is Correctly Specified

Next, we consider the case where the outcome regression is correctly specified so that $m_{0,t}^C(\mathcal{F}_{t^*-1}) = m_{0,t}^C(\mathcal{F}_{t^*-1};\mu_t^*)$, but the propensity score may not be correctly specified so that it may be the case that $p(\mathcal{F}_{t^*-1}) \neq p(\mathcal{F}_{t^*-1};\pi^*)$. In this case,

$$\begin{aligned}
&\mathbb{E}\left[\omega(D,\mathcal{F}_{t^*-1};\pi^*)(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1};\mu_t^*))\right] \\
&\qquad = \mathbb{E}\left[\omega(D,\mathcal{F}_{t^*-1};\pi^*)(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1}))|D=1\right]\mathbb{E}[D] \\
&\qquad + \mathbb{E}\left[\omega(D,\mathcal{F}_{t^*-1};\pi^*)(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1}))|D=0\right](1-\mathbb{E}[D]) \\
&\qquad := A + B
\end{aligned}$$

where the first equality holds by the law of iterated expectations. We consider Term A and Term B in turn next. Next, notice that

$$\begin{aligned}
A &= \mathbb{E}[(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1}))|D=1] \\
&= \mathbb{E}[C_t|D=1] - \mathbb{E}\left[\mathbb{E}[C_t(0)|\mathcal{F}_{t^*-1}, D=1]|D=1\right] \\
&= ATT_t^C
\end{aligned}$$

where the first equality holds by the definition of the weights (and by the expectation being conditional on $D=1$), the second equality holds by unconfoundedness (which holds here by Theorem 2), and the last equality holds by the law of iterated expectations and the definition of ATT_t^C.

Finally, consider Term B

$$\begin{aligned}
B &= \mathbb{E}\left[\omega(D,\mathcal{F}_{t^*-1};\pi^*)(\mathbb{E}[C_t|\mathcal{F}_{t^*-1}, D=0] - m_{0,t}^C(\mathcal{F}_{t^*-1}))|D=0\right](1-\mathbb{E}[D]) \\
&= 0
\end{aligned}$$

where the first equality holds by the law of iterated expectations and the last equality holds by the definition of $m_{0,t}^C(\mathcal{F}_{t^*-1})$.

Combining the results for Term A and Term B implies the result for this part when the outcome regression is correctly specified. □

Next, we provide additional results related to conducting inference. For this part, we primarily follow Sant'Anna and Zhao (2020) who consider similar doubly robust estimands to the ones considered in the current paper. Interestingly, that paper considers estimation and inference in a (conditional) difference in differences setup; however, most of their arguments apply to the current paper with only minor modifications.

Notice that, under Assumptions SA.1 to SA.3,

$$\sqrt{n}\left(\hat{\pi} - \pi^*\right) = Q_{ps}^{-1}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}\iota^{ps}(D_l, \mathcal{F}_{lt^*-1}) + o_p(1) \tag{SA.3}$$

$$\sqrt{n}\left(p(\mathcal{F}_{t^*-1};\hat{\pi}) - p(\mathcal{F}_{t^*-1};\pi^*)\right) = \kappa_{ps}(\mathcal{F}_{t^*-1})'Q_{ps}^{-1}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}\iota^{ps}(D_l, \mathcal{F}_{lt^*-1}) + o_p(1) \tag{SA.4}$$

where $\iota^{ps}(D, \mathcal{F}_{t^*-1}) = h_{ps}(\mathcal{F}_{t^*-1})(D - p(\mathcal{F}_{t^*-1}; \pi^*))$ and $\kappa_{ps}(\mathcal{F}_{t^*-1}) = \lambda(h_{ps}(\mathcal{F}_{t^*-1})'\pi^*)h_{ps}(\mathcal{F}_{t^*-1})$ with λ the derivative of Λ. Similarly,

$$\sqrt{n}\left(\hat{\mu}_t - \mu_t^*\right) = Q_{0,or}^{-1} \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \iota^{or}(C_{lt}, D_l, \mathcal{F}_{lt^*-1}) + o_p(1) \tag{SA.5}$$

$$\sqrt{n}\left(m_{0,t}^C(\mathcal{F}_{t^*-1}; \hat{\mu}_t) - m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*)\right) = h_{or}(\mathcal{F}_{t^*-1})' Q_{0,or}^{-1} \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \iota^{or}(C_{lt}, D_l, \mathcal{F}_{lt^*-1}) + o_p(1) \tag{SA.6}$$

where $\iota^{or}(C_t, D, \mathcal{F}_{t^*-1}) = \frac{(1-D)}{(1-\mathbb{E}[D])^{1/2}} h_{or}(\mathcal{F}_{t^*-1})(C_t - h_{or}(\mathcal{F}_{t^*-1})'\mu_t^*)$.

Next, we provide an asymptotically linear representation for estimating ATT_t^C as well as its limiting distribution. Before doing that, we introduce some additional notation. First, let $W_t := (C_t, D, \mathcal{F}_{t^*-1}')'$, $W_{lt} := (C_{lt}, D_l, \mathcal{F}_{lt^*-1}')'$, and $W := (W_{t^*}', \ldots, W_{\mathcal{T}}')'$. Further, define

$$\psi_{1,t}^A(W_{lt}) := -\frac{\mathbb{E}\left[D(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*)\right]}{\mathbb{E}[D]^2}(D_l - \mathbb{E}[D])$$

$$\psi_{0,t}^{B_1}(W_{lt}) := \frac{1}{\mathbb{E}\left[\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}; \pi^*)\right]} \mathbb{E}\left[\frac{(1-D)(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*))}{(1 - p(\mathcal{F}_{t^*-1}; \pi^*))^2} \kappa_{ps}(\mathcal{F}_{t^*-1})'\right] Q_{ps}^{-1} \iota^{ps}(D_l, \mathcal{F}_{lt^*-1})$$

$$\psi_{0,t}^{B_{21}}(W_{lt}) := \zeta_t \mathbb{E}\left[\left(\frac{(1-D)}{(1 - p(\mathcal{F}_{t^*-1}; \pi^*))^2}\right) \kappa_{ps}(\mathcal{F}_{t^*-1})'\right] Q_{ps}^{-1} \iota^{ps}(D_l, \mathcal{F}_{lt^*-1})$$

$$\psi_{0,t}^{B_{22}}(W_{lt}) := \zeta_t \left(\tilde{\omega}_0(D_l, \mathcal{F}_{lt^*-1}; \pi^*) - \mathbb{E}\left[\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}; \pi^*)\right]\right)$$

$$\psi_t^C(W_{lt}) := \mathbb{E}\left[\omega(D_l, \mathcal{F}_{lt^*-1}; \pi^*) h_{or}(\mathcal{F}_{lt^*-1})'\right] Q_{0,or}^{-1} \iota^{or}(C_{lt}, \mathcal{F}_{lt^*-1})$$

$$\psi_t^D(W_{lt}) := \omega(D_l, \mathcal{F}_{lt^*-1}; \pi^*)(C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \mu_t^*)) - ATT_t^C$$

and where

$$\zeta_t := \frac{\mathbb{E}\left[\omega_0(D, \mathcal{F}_{t^*-1}; \pi^*)(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*))\right]}{\mathbb{E}\left[\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}; \pi^*)\right]}$$

Next, define

$$\psi_t(W_{lt}) := \psi_{1,t}^A(W_{lt}) - \psi_{0,t}^{B_1}(W_{lt}) + \psi_{0,t}^{B_{21}}(W_{lt}) + \psi_{0,t}^{B_{22}}(W_{lt}) - \psi_t^C(W_{lt}) + \psi_t^D(W_{lt})$$

and $\Psi(W) := (\psi_{t^*}(W_{t^*}), \ldots, \psi_{\mathcal{T}}(W_{\mathcal{T}}))'$. Moreover, let $ATT^C = (ATT_{t^*}^C, \ldots, ATT_{\mathcal{T}}^C)'$, and, likewise, $\widehat{ATT}^C = (\widehat{ATT}_{t^*}^C, \ldots, \widehat{ATT}_{\mathcal{T}}^C)'$.

Proposition SA.2. *Under the Stochastic SIRD Model for Untreated Potential Outcomes and Assumptions 1 and SA.1 to SA.4, for any $t^* \leq t \leq \mathcal{T}$,*

$$\sqrt{n}(\widehat{ATT}_t^C - ATT_t^C) = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \psi_t(W_{lt}) + o_p(1)$$

In addition,

$$\sqrt{n}(\widehat{ATT}^C - ATT^C) \xrightarrow{d} N(0, V)$$

where $V = \mathbb{E}[\Psi(W)\Psi(W)']$.

Before proving Proposition SA.2, we provide an additional helpful result that is used in the proof.

Lemma SA.1. *Under the Stochastic SIRD Model for Untreated Potential Outcomes and Assumptions 1 and SA.1 to SA.4,*

$$\sqrt{n}\left(\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}, \hat{\pi}) - \tilde{\omega}_0(D, \mathcal{F}_{t^*-1}, \pi^*)\right) = \left(\frac{(1-D)}{(1-p(\mathcal{F}_{t^*-1}; \pi^*))^2}\right)\sqrt{n}\left(p(\mathcal{F}_{t^*-1}; \hat{\pi}) - p(\mathcal{F}_{t^*-1}; \pi^*)\right) + o_p(1)$$

Proof. First, notice that

$$\begin{aligned}
&\sqrt{n}\left(\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}, \hat{\pi}) - \tilde{\omega}_0(D, \mathcal{F}_{t^*-1}, \pi^*)\right) \\
&\qquad = \sqrt{n}\left(\frac{p(\mathcal{F}_{t^*-1}; \hat{\pi})}{(1-p(\mathcal{F}_{t^*-1}; \hat{\pi}))} - \frac{p(\mathcal{F}_{t^*-1}; \pi^*)}{(1-p(\mathcal{F}_{t^*-1}; \pi^*))}\right)(1-D) \\
&\qquad = \left(\frac{(1-D)}{(1-p(\mathcal{F}_{t^*-1}; \hat{\pi}))(1-p(\mathcal{F}_{t^*-1}; \pi^*))}\right)\sqrt{n}\left(p(\mathcal{F}_{t^*-1}; \hat{\pi}) - p(\mathcal{F}_{t^*-1}; \pi^*)\right) \\
&\qquad = \left(\frac{(1-D)}{(1-p(\mathcal{F}_{t^*-1}; \pi^*))^2}\right)\sqrt{n}\left(p(\mathcal{F}_{t^*-1}; \hat{\pi}) - p(\mathcal{F}_{t^*-1}; \pi^*)\right) + o_p(1)
\end{aligned}$$

where the first equality holds by the definition of $\tilde{\omega}_0$, the second equality by cross-multiplying and rearranging, and the last equality by the weak law of large numbers and the continuous mapping theorem. □

Proof of Proposition SA.2. By adding and subtracting terms (and using the result from Proposition SA.1), we can write

$$\begin{aligned}
\sqrt{n}\left(\widehat{ATT}_t^C - ATT_t^C\right) &= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}(\hat{\omega}_1(D_l) - \omega_1(D_l))(C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \hat{\mu}_t)) \\
&\quad - \frac{1}{\sqrt{n}}\sum_{l=1}^{n}(\hat{\omega}_0(D_l, \mathcal{F}_{lt^*-1}; \hat{\pi})) - \omega_0(D_l, \mathcal{F}_{lt^*-1}; \pi^*))(C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \hat{\mu}_t)) \\
&\quad - \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\omega(D_l, \mathcal{F}_{lt^*-1}; \pi^*)(m_{0,t}^C(\mathcal{F}_{lt^*-1}; \hat{\mu}_t) - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \mu_t^*)) \\
&\quad + \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\omega(D_l, \mathcal{F}_{lt^*-1}; \pi^*)(C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \mu_t^*)) - ATT_t^C \\
&:= A - B - C + D
\end{aligned}$$

Term A involves the estimation effect of the first component of the weights, Term B involves the estimation effect of the second component of the weights, Term C involves the estimation effect of the outcome regression, and Term D is the estimation effect if the weights and outcome regression were known. We consider each of these terms in turn next.

First, we consider Term A. Notice that we can re-write it as

$$
\begin{aligned}
A &= \frac{1}{\sqrt{n}} \sum_{l=1}^{n} (\hat{\omega}_1(D_l) - \omega_1(D_l))(C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \mu_t^*)) \\
&- \frac{1}{\sqrt{n}} \sum_{l=1}^{n} (\hat{\omega}_1(D_l) - \omega_1(D_l))(m_{0,t}^C(\mathcal{F}_{lt^*-1}; \hat{\mu}_t) - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \mu_t^*)) \\
&= A_1 - A_2
\end{aligned}
$$

Now consider Term A_1.

$$
\begin{aligned}
A_1 &= \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \left(\frac{D_l}{\bar{D}} - \frac{D_l}{\mathbb{E}[D]} \right) (C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \mu_t^*)) \\
&= -\frac{1}{\sqrt{n}} \sum_{l=1}^{n} D_l \frac{(\bar{D} - \mathbb{E}[D])}{\bar{D}\mathbb{E}[D]} (C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \mu_t^*)) \\
&= -\frac{1}{\sqrt{n}} \sum_{l=1}^{n} \frac{\mathbb{E}\left[D(C_t - m_{0,t}^C(\mathcal{F}_{t^*-1}; \mu_t^*)\right]}{\mathbb{E}[D]^2} (D_l - \mathbb{E}[D]) + o_p(1) \\
&= \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \psi_{1,t}^A(W_{lt}) + o_p(1)
\end{aligned}
$$

where the first equality holds by the definition of the weights, the second equality holds by combining terms, the third equality holds by the weak law of large numbers and continuous mapping theorem, and the last equality by the definition of $\psi_{1,t}^A$.

Next, consider Term A_2,

$$
\begin{aligned}
A_2 &= \frac{1}{n} \sum_{l=1}^{n} \left(\frac{D_l}{\bar{D}} - \frac{D_l}{\mathbb{E}[D]} \right) h_{or}(\mathcal{F}_{lt^*-1})' \sqrt{n}(\hat{\mu}_t - \mu_t^*) + o_p(1) \\
&= -\frac{1}{n} \sum_{l=1}^{n} \left(\frac{D_l(\bar{D} - \mathbb{E}[D])}{\bar{D}\mathbb{E}[D]} \right) h_{or}(\mathcal{F}_{lt^*-1})' \sqrt{n}(\hat{\mu}_t - \mu_t^*) + o_p(1) \\
&= -\frac{\mathbb{E}[D h_{or}(\mathcal{F}_{t^*-1})']}{\mathbb{E}[D]^2} (\bar{D} - \mathbb{E}[D]) \sqrt{n}(\hat{\mu}_t - \mu_t^*) + o_p(1) \\
&= o_p(1)
\end{aligned}
$$

where the first equality holds from the definitions of the weights and by Equation (SA.6), the second equality holds by combining terms, the third equality by the weak law of large numbers and continuous mapping theorem, and the last equality holds because $(\bar{D} - \mathbb{E}[D]) = o_p(1)$ and $\sqrt{n}(\hat{\mu}_t - \mu_t^*) = O_p(1)$.

Next, consider Term B,

$$
\begin{aligned}
B &= \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \left(\frac{\tilde{\omega}_0(D_l, \mathcal{F}_{lt^*-1}; \hat{\pi}) - \tilde{\omega}_0(D_l, \mathcal{F}_{lt^*-1}; \pi^*)}{\frac{1}{n}\sum_{h=1}^{n} \tilde{\omega}_0(D_h, \mathcal{F}_{ht^*-1}; \hat{\pi})} \right) (C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \hat{\mu}_t)) \\
&+ \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \left(\frac{\tilde{\omega}_0(D_l, \mathcal{F}_{lt^*-1}; \pi^*)}{\frac{1}{n}\sum_{h=1}^{n} \tilde{\omega}_0(D_h, \mathcal{F}_{ht^*-1}; \hat{\pi})} - \frac{\tilde{\omega}_0(D_l, \mathcal{F}_{lt^*-1}; \pi^*)}{\mathbb{E}\left[\tilde{\omega}_0(D, \mathcal{F}_{t^*-1}; \pi^*)\right]} \right) (C_{lt} - m_{0,t}^C(\mathcal{F}_{lt^*-1}; \hat{\mu}_t)) \\
&:= B_1 + B_2
\end{aligned}
$$

For Term B_1,

$$\begin{aligned}
B_1 &= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\left(\frac{\tilde{\omega}_0(D_l,\mathcal{F}_{lt^*-1};\hat{\pi})-\tilde{\omega}_0(D_l,\mathcal{F}_{lt^*-1};\pi^*)}{\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]}\right)(C_{lt}-m^C_{0,t}(\mathcal{F}_{lt^*-1};\mu^*_t))+o_p(1)\\
&= \frac{1}{n}\sum_{l=1}^{n}\left(\frac{(1-D_l)(C_{lt}-m^C_{0,t}(\mathcal{F}_{lt^*-1};\mu^*_t))}{(1-p(\mathcal{F}_{lt^*-1};\pi^*))^2\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]}\right)\sqrt{n}\left(p(\mathcal{F}_{lt^*-1};\hat{\pi})-p(\mathcal{F}_{lt^*-1};\pi^*)\right)+o_p(1)\\
&= \frac{1}{\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]}\mathbb{E}\left[\frac{(1-D)(C_t-m^C_{0,t}(\mathcal{F}_{t^*-1};\mu^*_t))}{(1-p(\mathcal{F}_{t^*-1};\pi^*))^2}\kappa_{ps}(\mathcal{F}_{t^*-1})'\right]Q_{ps}^{-1}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}\iota^{ps}(D_l,\mathcal{F}_{lt^*-1})+o_p(1)\\
&= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\psi^{B_1}_{0,t}(W_{lt})+o_p(1)
\end{aligned}$$

where the first equality follows from similar arguments as above, the second equality uses Lemma SA.1, and the last equality holds from Equation (SA.4) and by the weak law of large numbers and continuous mapping theorem.

Next, for term B_2,

$$\begin{aligned}
B_2 &= -\frac{1}{n}\sum_{l=1}^{n}\frac{\omega_0(D_l,\mathcal{F}_{lt^*-1};\pi^*)(C_{lt}-m^C_{0,t}(\mathcal{F}_{lt^*-1};\hat{\mu}_t))}{\frac{1}{n}\sum_{h=1}^{n}\tilde{\omega}_0(D_h,\mathcal{F}_{ht^*-1};\hat{\pi})}\left(\frac{1}{\sqrt{n}}\sum_{h=1}^{n}\tilde{\omega}_0(D_h,\mathcal{F}_{ht^*-1};\hat{\pi})-\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]\right)\\
&= -\frac{\mathbb{E}\left[\omega_0(D,\mathcal{F}_{t^*-1};\pi^*)(C_t-m^C_{0,t}(\mathcal{F}_{t^*-1};\mu^*_t))\right]}{\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]}\left(\frac{1}{\sqrt{n}}\sum_{h=1}^{n}\tilde{\omega}_0(D_h,\mathcal{F}_{ht^*-1};\hat{\pi})-\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]\right)+o_p(1)\\
&= -\frac{\mathbb{E}\left[\omega_0(D,\mathcal{F}_{t^*-1};\pi^*)(C_t-m^C_{0,t}(\mathcal{F}_{t^*-1};\mu^*_t))\right]}{\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]}\left(\frac{1}{\sqrt{n}}\sum_{h=1}^{n}\tilde{\omega}_0(D_h,\mathcal{F}_{ht^*-1};\hat{\pi})-\tilde{\omega}_0(D_h,\mathcal{F}_{ht^*-1};\pi^*)\right)\\
&\quad -\frac{\mathbb{E}\left[\omega_0(D,\mathcal{F}_{t^*-1};\pi^*)(C_t-m^C_{0,t}(\mathcal{F}_{t^*-1};\mu^*_t))\right]}{\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]}\left(\frac{1}{\sqrt{n}}\sum_{h=1}^{n}\tilde{\omega}_0(D_h,\mathcal{F}_{ht^*-1};\pi^*)-\mathbb{E}\left[\tilde{\omega}_0(D,\mathcal{F}_{t^*-1};\pi^*)\right]\right)+o_p(1)\\
&:= -B_{21}-B_{22}
\end{aligned}$$

where the first equality holds by cross-multiplying and the definition of ω_0, the second equality holds by the weak law of large numbers and the continuous mapping theorem, and the third equality by adding and subtracting terms.

For B_{21}, notice that

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{h=1}^{n}\tilde{\omega}_0(D_h,\mathcal{F}_{ht^*-1};\hat{\pi})-\tilde{\omega}_0(D_h,\mathcal{F}_{ht^*-1};\pi^*)\\
&\qquad= \frac{1}{n}\sum_{h=1}^{n}\left(\frac{(1-D_h)}{(1-p(\mathcal{F}_{ht^*-1};\pi^*))^2}\right)\sqrt{n}\left(p(\mathcal{F}_{ht^*-1};\hat{\pi})-p(\mathcal{F}_{ht^*-1};\pi^*)\right)+o_p(1)\\
&\qquad= \frac{1}{n}\sum_{h=1}^{n}\left(\frac{(1-D_h)}{(1-p(\mathcal{F}_{ht^*-1};\pi^*))^2}\right)\kappa_{ps}(\mathcal{F}_{ht^*-1})'Q_{ps}^{-1}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}\iota^{ps}(D_l,\mathcal{F}_{lt^*-1})+o_p(1)\\
&\qquad= \mathbb{E}\left[\left(\frac{(1-D)}{(1-p(\mathcal{F}_{t^*-1};\pi^*))^2}\right)\kappa_{ps}(\mathcal{F}_{t^*-1})'\right]Q_{ps}^{-1}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}\iota^{ps}(D_l,\mathcal{F}_{lt^*-1})+o_p(1)
\end{aligned}$$

where the first equality holds by Lemma SA.1, the second equality holds by Equation (SA.4), and the last equality holds by the weak law of large numbers and continuous mapping theorem. This implies that

$$B_{21} = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \psi_{0,t}^{B_{21}}(W_{lt}) + o_p(1)$$

For B_{22}, notice that it is immediately given by

$$B_{22} = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \psi_{0,t}^{B_{22}}(W_{lt})$$

Now, we turn to Term C.

$$\begin{aligned} C &= \frac{1}{n} \sum_{l=1}^{n} \omega(D_l, \mathcal{F}_{lt^*-1}; \pi^*) h_{or}(\mathcal{F}_{lt^*-1})' Q_{0,or}^{-1} \frac{1}{\sqrt{n}} \sum_{h=1}^{n} \iota^{or}(C_{ht}, D_h, \mathcal{F}_{ht^*-1}) + o_p(1) \\ &= \mathbb{E}\left[\omega(D, \mathcal{F}_{t^*-1}; \pi^*) h_{or}(\mathcal{F}_{t^*-1})'\right] Q_{0,or}^{-1} \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \iota^{or}(C_{lt}, D_l, \mathcal{F}_{lt^*-1}) + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \psi_t^C(W_{lt}) + o_p(1) \end{aligned}$$

where the first equality holds by Equation (SA.6) and the second equality by the weak law of large numbers and continuous mapping theorem.

Finally, for Term D, notice that it is immediately given by

$$D = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} \psi_t^D(W_{lt})$$

Combining the results for Terms A-D establishes the first part of the result. Asymptotic normality holds by applying the central limit theorem jointly for $t = t^*, \ldots, \mathcal{T}$.

□

In order to actually conduct inference, we use the multiplier bootstrap. To start with, we describe the multiplier bootstrap procedure that we use. First, define $\hat{\Psi}$ as the sample analogue of Ψ. Next, let ξ denote an n-dimensional vector of iid random variables with mean zero, variance one, finite third moment, and that is independent of the original data (e.g., two common choices for ξ are either iid draws from $N(0,1)$ or to draw ξ equal to -1 or 1 each with probability $1/2$). Then, we consider a bootstrapped version of $\widehat{ATT}^C$ given by

$$\widehat{ATT}^{C,*} = \widehat{ATT}^C + \frac{1}{n} \sum_{l=1}^{n} \xi_l \hat{\Psi}(W_l)$$

Relative to the more common nonparametric bootstrap, there are two main advantages of the multiplier bootstrap. First, it is very fast to compute as it essentially only involves making random draws from a simple distribution rather than re-estimating $\widehat{ATT}^C$ at every bootstrap iteration. Second, since the multiplier bootstrap perturbs the influence function rather than re-drawing data, this approach does not run into the practical problem of particular bootstrap iterations not being able to estimate the parameters of interest (e.g., this can occur when there are discrete covariates where some combinations occur infrequently).

The next result shows that the proposed multiplier bootstrap procedure follows the same limiting distribution as the original estimator of ATT^C.

Proposition SA.3. *Under the Stochastic SIRD Model for Untreated Potential Outcomes and Assumptions 1 and SA.1 to SA.4,*

$$\sqrt{n}(\widehat{ATT}^{C,*} - \widehat{ATT}^{C}) \xrightarrow{d^*} N(0, V)$$

where V is the same as in Proposition SA.2 and $\xrightarrow{d^*}$ denotes convergence in bootstrap distribution.

Proof. Given the results in Proposition SA.2, the proof of Proposition SA.3 follows from the same arguments as in the proof of Theorem 3 in Callaway and Sant'Anna (2020). □

Recall that $ATT^C = (ATT_{t^*}, \ldots, ATT^C_{\mathcal{T}})'$ so that the results in Propositions SA.2 and SA.3 hold jointly across post-treatment periods. It is therefore straightforward to construct uniform confidence bands that asymptotically cover ATT^C simultaneously with fixed probability $1-\alpha$. In particular, one can construct a uniform confidence band as follows.

Algorithm SA.1 (Multiplier Bootstrap for Uniform Confidence Band)**.**

Step 1: Draw ξ_l, $l = 1, \ldots, n$ which are iid across l, have mean 0, variance 1, and finite third moment

Step 2: Set $\widehat{ATT}^{C,*} = \widehat{ATT}^{C} + \frac{1}{n}\sum_{l=1}^{n} \xi_l \hat{\Psi}(W_l)$.

Step 3: For $t = t^*, \ldots, \mathcal{T}$, compute $\hat{R}^*_t = \sqrt{n}(\widehat{ATT}^{C,*}_t - \widehat{ATT}^{C}_t)$ where $\widehat{ATT}^{C,*}_t$ is a particular element of $\widehat{ATT}^{C,*}$ from Step 2.

Repeat Steps 1-3 B times where B is the (large) number of bootstrap iterations.

Step 4: Compute $\hat{V}_t^{1/2} = (q_{0.75,t} - q_{0.25,t})/(z_{0.75} - z_{0.25})$ where $q_{p,t}$ is the pth quantile of $\hat{R}^*_t$ across the B bootstrap iterations and z_p is the pth quantile of the standard normal distribution.

Step 5: For each bootstrap draw, compute sup-t $= \max_{t \in \{t^*, \ldots, \mathcal{T}\}} |\hat{R}^*_t| \hat{V}_t^{-1/2}$

Step 6: Construct the critical value $\hat{c}_{1-\alpha}$ as the $(1-\alpha)$ quantile of the B bootstrap draws of sup-t.

Step 7: Construct the uniform confidence band $\hat{C}_t = [\widehat{ATT}^{C}_t \pm \hat{c}_{1-\alpha} \hat{V}_t^{-1/2}/\sqrt{n}]$

The next result shows that the uniform confidence band from Algorithm SA.1 has the asymptotically correct coverage.

Proposition SA.4. *Under the Stochastic SIRD Model for Untreated Potential Outcomes and Assumptions 1 and SA.1 to SA.4,*

$$P(ATT^C_t \in \hat{C}_t \ \ \textit{for all } t \in \{t^*, \ldots, \mathcal{T}\}) \to 1 - \alpha$$

as $n \to \infty$.

Proof. Given the result in Proposition SA.3, the result holds by the same argument as for Theorem 3 (and Corollary 1) in Callaway and Sant'Anna (2020). □

To conclude, we provide the limiting distribution and briefly discuss an inference procedure for the adjusted regression DID estimator of ATT^Y discussed in the main text. The arguments only require a minor extension of the results for ATT^C given above; therefore, we only briefly sketch the additional arguments for adjusted regression DID here.

For the arguments below, suppose that the Stochastic SIRD Model for Untreated Potential Outcomes, Assumption 1, Assumption SA.1, assumptions analogous to Assumptions SA.2 to SA.4 but for current Covid-19 cases rather than cumulative cases all hold. In addition, for all $t = t^*, \ldots, \mathcal{T}$, we assume that $Q_t^{0,I} := \mathbb{E}[(1, \Delta^{(t^*-1,t)} I_t)'(1, \Delta^{(t^*-1,t)} I_t)|D = 0]$ is positive definite and that $\mathbb{E}[\Delta^{(t^*-1,t)} Y_t^4 | D = d] < \infty$ for $d \in \{0, 1\}$.

We also define

$$\psi^A_{Y,t}(W_{lt}) := -\left(\frac{\mathbb{E}[D\Delta^{(t^*-1,t)}Y_t]}{\mathbb{E}[D]^2}(D_l - \mathbb{E}[D])\right) + \left(\frac{D_l}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}Y_{lt} - \mathbb{E}\left[\frac{D}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}Y_t\right]\right)$$

$$\psi^B_{Y,t}(W_{lt}) := \eta_{\tau,\alpha}(D_l, \Delta^{(t^*-1,t)}I_{lt}, \Delta^{(t^*-1,t)}Y_{lt})'(1, \mathbb{E}[\Delta^{(t^*-1,t)}I_t(0)|D = 1])'$$

$$\psi^C_{Y,t}(W_l t) := \alpha\left\{-\left(\frac{\mathbb{E}[D\Delta^{(t^*-1,t)}I_t]}{\mathbb{E}[D]^2}(D_l - \mathbb{E}[D])\right) + \left(\frac{D_l}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}I_{lt} - \mathbb{E}\left[\frac{D}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}I_t\right]\right) - \psi_{I,t}(W_{lt})\right\}$$

where $\eta_{\tau,\alpha}(D, \Delta^{(t^*-1,t)}I_t, \Delta^{(t^*-1,t)}Y_t)$ is defined below and where ψ_t^I is the same as ψ_t in Proposition SA.2 except with I (the current number of Covid-19 cases) replacing C (the cumulative number of Covid-19 cases) everywhere. Next, define

$$\psi_{Y,t}(W_{lt}) := \psi^A_{Y,t}(W_{lt}) - \psi^B_{Y,t}(W_{lt}) - \psi^C_{Y,t}(W_{lt})$$

and $\Psi_Y(W) = (\psi_{Y,t^*}(W_{t^*}), \ldots, \psi_{Y,\mathcal{T}}(W_{\mathcal{T}}))'$. Define $ATT^Y = (ATT^Y_{t^*}, \ldots, ATT^Y_{\mathcal{T}})'$ and $\widehat{ATT}^Y = (\widehat{ATT}^Y_{t^*}, \ldots, \widehat{ATT}^Y_{\mathcal{T}})'$. Then, the following results all hold using essentially the same arguments as above:

$$\sqrt{n}(\widehat{ATT}^Y_t - \widehat{ATT}^Y_t) = \frac{1}{\sqrt{n}}\sum_{l=1}^n \psi_{Y,t}(W_{lt}) + o_p(1)$$

Moreover,

$$\sqrt{n}(\widehat{ATT}^Y - ATT^Y) \xrightarrow{d} N(0, V_Y)$$

where $V_Y = \mathbb{E}[\Psi_Y(W)\Psi_Y(W)']$. In addition, the multiplier bootstrap can be used to conduct inference and uniform confidence bands can be constructed analogously as above. We show the result for the influence function below. The remaining results hold immediately from the same arguments as above given the distinct expression for the influence function in this case.

Start by noticing that

$$\begin{aligned}
\sqrt{n}(\widehat{ATT}^Y_t - ATT^Y_t) &= \frac{1}{\sqrt{n}}\sum_{l=1}^n \left(\hat{\omega}_1(D_l)\Delta^{(t^*-1,t)}Y_{lt} - \mathbb{E}\left[\frac{D}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}Y_t\right]\right) \\
&\quad - \sqrt{n}\left((\hat{\tilde{\tau}}_t - \tilde{\tau}_t) + (\hat{\alpha} - \alpha)\hat{\mathbb{E}}[\Delta^{(t^*-1,t)}I_t(0)|D = 1]\right) \\
&\quad - \alpha\sqrt{n}\left(\hat{\mathbb{E}}[\Delta^{(t^*-1,t)}I_t(0)|D = 1] - \mathbb{E}[\Delta^{(t^*-1,t)}I_t(0)|D = 1]\right) \\
&:= A - B - C
\end{aligned}$$

where

$$\hat{\mathbb{E}}[\Delta^{(t^*-1,t)} I_t(0)|D=1] = \frac{1}{n}\sum_{l=1}^{n}\left(\frac{D_l}{\bar{D}}\Delta^{(t^*-1,t)} I_{lt} - \hat{\omega}(D_l,\mathcal{F}_{lt^*-1})(I_{lt} - \hat{m}^C_{0,t}(\mathcal{F}_{lt^*-1}))\right)$$

Following similar arguments as above, it follows that

$$\begin{aligned}
A &= -\frac{\mathbb{E}[D\Delta^{(t^*-1,t)}Y_t]}{\mathbb{E}[D]^2}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}(D_l - \mathbb{E}[D]) + \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\left(\frac{D_l}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}Y_{lt} - \mathbb{E}\left[\frac{D}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}Y_t\right]\right) + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\psi^A_{Y,t}(W_{lt}) + o_p(1)
\end{aligned}$$

For term B, first notice that

$$\begin{aligned}
\sqrt{n}\begin{pmatrix}\hat{\tilde{\tau}}_t - \tilde{\tau}_t \\ \hat{\alpha} - \alpha\end{pmatrix} &= Q_t^{0,I^{-1}}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}\frac{(1-D_l)}{(1-\mathbb{E}[D])^{1/2}}(1,\Delta^{(t^*-1,t)}I_{lt})'\Delta^{(t^*-1,t)}Y_{lt} + o_p(1) \\
&:= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\eta_{\tau,\alpha}(D_l,\Delta^{(t^*-1,t)}I_{lt},\Delta^{(t^*-1,t)}Y_{lt}) + o_p(1)
\end{aligned}$$

Thus,

$$\begin{aligned}
B &= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\eta_{\tau,\alpha}(D_l,\Delta^{(t^*-1,t)}I_{lt},\Delta^{(t^*-1,t)}Y_{lt})'(1,\mathbb{E}[\Delta^{(t^*-1,t)}I_t(0)|D=1])' + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\psi^B_{Y,t}(W_{lt}) + o_p(1)
\end{aligned}$$

For Term C, first notice that

$$\begin{aligned}
&\sqrt{n}\left(\hat{\mathbb{E}}[\Delta^{(t^*-1,t)}I_t(0)|D=1] - \mathbb{E}[\Delta^{(t^*-1,t)}I_t(0)|D=1]\right) \\
&\quad= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\left(\hat{\omega}_1(D_l)\Delta^{(t^*-1,t)}I_{lt} - \mathbb{E}\left[\frac{D}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}I_t\right]\right) \\
&\quad- \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\left(\hat{\omega}(D_l,\mathcal{F}_{lt^*-1})(I_{lt} - \hat{m}^I_{0,t}(\mathcal{F}_{lt^*-1})) - \mathbb{E}\left[\omega(D,\mathcal{F}_{t^*-1})(I_t - m^I_{0,t}(\mathcal{F}_{t^*-1}))\right]\right) \\
&\quad:= C_1 - C_2
\end{aligned}$$

Using the same arguments as above, it immediately follows that

$$C_1 = -\frac{\mathbb{E}[D\Delta^{(t^*-1,t)}I_t]}{\mathbb{E}[D]^2}\frac{1}{\sqrt{n}}\sum_{l=1}^{n}(D_l - \mathbb{E}[D]) + \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\left(\frac{D_l}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}I_{lt} - \mathbb{E}\left[\frac{D}{\mathbb{E}[D]}\Delta^{(t^*-1,t)}I_t\right]\right)$$

For C_2, notice that it is exactly the same (up to I replacing C) as in Proposition SA.2, which implies that

$$C_2 = \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\psi^I_t(W_{lt}) + o_p(1)$$

Multiplying the expressions for C_1 and C_2 by α implies that

$$C = \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\psi^C_{Y,t}(W_{lt}) + o_p(1)$$

Plugging back in the expressions for Terms A, B, and C provides the influence function.

References

[1] Callaway, Brantly and Pedro HC Sant'Anna. "Difference-in-differences with multiple time periods". Working Paper. 2020.

[2] Sant'Anna, Pedro HC and Jun Zhao. "Doubly robust difference-in-differences estimators". *Journal of Econometrics* 219.1 (2020), pp. 101–122.