# Supplementary Appendix

# Distributional Effects with Two-Sided Measurement Error: An Application to Intergenerational Income Mobility*

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This Supplementary Appendix contains proofs for several of the results from the main text as well as some supplementary results. Appendix SA contains some additional theoretical results as well as proofs for some of the results provided in the main text. Appendix SB provides conditions under which our main target parameters have a causal interpretation. Appendix SC explains how to extend our results in the presence of life-cycle measurement error. Appendix SD provides additional results for our application on intergenerational mobility. Finally, Appendix SE provides additional Monte Carlo simulations with Laplace measurement error and with correlated measurement error.

## SA Additional Theoretical Results and Proofs

The first part of this section contains the proof of Lemma 1 from the main text and then states and proves Lemma S1, which is used in the proof of Theorem 4. The second part of this section proves parts (b)-(f) of Theorem 4.

### SA.1 Useful Lemmas

***Proof of Lemma 1.*** By the triangle inequality:

$$\begin{aligned}
&\Big|Q_{n,\tau}\big(\beta(\tau),\sigma\big) - Q_\tau\big(\beta(\tau),\sigma\big)\Big| \\
&\leq \frac{1}{n}\sum_{i=1}^n \Big| \int_{\mathcal{U}} \rho_\tau\left(Y_i - u - X_i'\beta(\tau)\right)\Big(\widehat{f}^{(S)}_{U_{Y^*}|Y,X}(u \mid Y_i, X_i;\sigma) - f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i;\sigma)\Big) du\Big| \\
&+ \Big|\frac{1}{n}\sum_{i=1}^n \Big\{ \int_{\mathcal{U}} \rho_\tau\left(Y_i - u - X_i'\beta(\tau)\right) f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i;\sigma)\, du
\end{aligned}$$

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$$- \mathbb{E}\Big[ \int_{\mathcal{U}} \rho_\tau \left(Y_i - u - X_i'\beta\right) f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\, du\Big] \Big\} \Big|.$$

Define $\mathcal{E}_\tau := Y_i - X_i'\beta(\tau)$. Note that $\rho_\tau(w) \le |w|$ and $\rho_\tau(w)$ is 1-Lipschitz.

First, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\Big| \int_{\mathcal{U}} \rho_\tau \left(Y_i - u - X_i'\beta(\tau)\right) \Big(\widehat{f}^{(S)}_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma) - f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\Big) du \Big| \\
&= \Big| \int_{\mathcal{U}} \left(\rho_\tau\left(\mathcal{E}_\tau - u\right) - \rho_\tau(\mathcal{E}_\tau) + \rho_\tau(\mathcal{E}_\tau)\right) \Big(\widehat{f}^{(S)}_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma) - f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\Big) du \Big| \\
&\le \Big| \int_{\mathcal{U}} \left(\rho_\tau\left(\mathcal{E}_\tau - u\right) - \rho_\tau(\mathcal{E}_\tau)\right) \cdot \Big(\widehat{f}^{(S)}_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma) - f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\Big) du \Big| \\
&+ \rho_\tau(\mathcal{E}_\tau) \Big| \int_{\mathcal{U}} \Big(\widehat{f}^{(S)}_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma) - f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\Big) du \Big| \\
&= \Big| \int_{\mathcal{U}} \left(\rho_\tau\left(\mathcal{E}_\tau - u\right) - \rho_\tau(\mathcal{E}_\tau)\right) \cdot \Big(\widehat{f}^{(S)}_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma) - f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\Big) du \Big|
\end{aligned}$$

since $\int_{\mathcal{U}} \Big(\widehat{f}^{(S)}_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma) - f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\Big) du = 0$. Let $\phi_\tau(u) := \rho_\tau(\mathcal{E}_\tau - u) - \rho_\tau(\mathcal{E}_\tau)$, and observe that since $\rho_\tau$ is 1–Lipschitz, $|\phi_\tau(u)| \le |u|$; this upper bound holds uniformly in $\tau \in (0, 1)$. Further, under Assumption 10(c), $\mathbb{E}[\phi^2_\tau(U(\sigma))|Y, X] \le \int_{\mathcal{U}} u^2 f_{U_{Y^*}|Y,X}(u \mid Y, X; \sigma) \le C\, a.s.$ uniformly in $\sigma \in \Gamma_\sigma$. Under Assumption 9,

$$\begin{aligned}
&\int_{\mathcal{U}} \left(\rho_\tau\left(\mathcal{E}_\tau - u\right) - \rho_\tau(\mathcal{E}_\tau)\right) \cdot \Big(\widehat{f}^{(S)}_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma) - f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\Big) du \\
&\qquad = \frac{1}{S} \sum_{s=1}^{S} \left(\phi_\tau(U_{is}(\sigma)) - \mathbb{E}[\phi_\tau(U_i(\sigma)) \mid Y_i, X_i]\right) = O_p(S^{-1/2}) = o_p(n^{-1/2})
\end{aligned}$$

by Prakasa Rao (2009, Theorem 10, eqn. 64), Chebyshev's inequality, that a $\beta$-mixing sequence is strongly mixing, and the condition $n/S = o(1)$.

Second, since $\rho_\tau$ is 1-Lipschitz, it follows from the triangle inequality, the Lyapunov inequality, the Schwarz inequality, Assumption 8, and Assumption 10 that

$$\begin{aligned}
\mathbb{E}\Big[ \int_{\mathcal{U}} \rho_\tau \left(Y - u - X'\beta(\tau)\right) f_{U_{Y^*}|Y,X}(u \mid Y, X; \sigma)\, du\Big] &= \mathbb{E}\Big[\mathbb{E}\big[\rho_\tau\big(\mathcal{E}_\tau - U(\sigma)\big) \mid Y, X\big]\Big] \\
&\le \sup_{\tau \in \mathcal{T}} \mathbb{E}[|Y - X'\beta(\tau)|] + \mathbb{E}\big[\mathbb{E}[(|U(\sigma)|)|Y, X]\big] \\
&\le \mathbb{E}[|Y|] + \mathbb{E}[\|X\|] \cdot \sup_{\tau \in \mathcal{T}} \|\beta(\tau)\| + \mathbb{E}\big[\left(\mathbb{E}[U(\sigma)^2|Y, X]\right)^{1/2}\big] \\
&< \infty.
\end{aligned}$$

Conclude under Assumption 4 and the strong law of large numbers that

$$\begin{aligned}
&\frac{1}{n} \sum_{i=1}^{n} \Big\{ \int_{\mathcal{U}} \rho_\tau \left(Y_i - u - X_i'\beta(\tau)\right) f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\, du \\
&\qquad - \mathbb{E}\Big[ \int_{\mathcal{U}} \rho_\tau \left(Y_i - u - X_i'\beta\right) f_{U_{Y^*}|Y,X}(u \mid Y_i, X_i; \sigma)\, du\Big] \Big\} \xrightarrow{a.s.} 0.
\end{aligned}$$

Combining both parts above concludes the proof of the assertion as claimed.

□

**Lemma S1.** *Under Assumptions 1 to 5 and 7 to 10,* $\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) - F_{Y^*}^{-1}(r)\big)$ *has the following representation:*

$$\begin{aligned}
&\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) - F_{Y^*}^{-1}(r)\big) \\
&\quad = -\frac{1}{f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))}\mathbb{E}\big[\mathbb{M}_{FY}^{L}\big(\mathrm{F}_{Y^*}^{-1}(r), X\big)\big]'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}_{Y^*}^{L} - \boldsymbol{\beta}_{Y^*}^{L} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*}\end{bmatrix} \\
&\qquad - \frac{1}{f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) - \mathbb{E}[\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)]\big) + o_p(1).
\end{aligned}$$

*Proof.* Let $y = \mathrm{F}_{Y^*}^{-1}(r)$, then $\widehat{\mathrm{F}}_{Y^*}(\widehat{y}) = r = \mathrm{F}_{Y^*}(y)$, then by the MVT, $\widehat{\mathrm{F}}_{Y^*}(\widehat{y}) - \mathrm{F}_{Y^*}(y) = 0 = \widehat{\mathrm{F}}_{Y^*}(y) - \mathrm{F}_{Y^*}(y) + \widehat{f}_{Y^*}(\bar{y})(\widehat{y} - y)$. Thus, by the Law of Iterated Expectations (LIE),

$$\begin{aligned}
&\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) - \mathrm{F}_{Y^*}^{-1}(r)\big) = \sqrt{n}(\widehat{y} - y) \\
&\quad = -\frac{1}{\widehat{f}_{Y^*}(\bar{y})}\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*}(y) - \mathrm{F}_{Y^*}(y)\big) \\
&\quad = -\frac{1}{\widehat{f}_{Y^*}(\bar{y})}\frac{1}{n}\sum_{i=1}^{n}\Big\{\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|X}(y \mid X_i) - \mathrm{F}_{Y^*|X}(y \mid X_i)\big) + \sqrt{n}\big(\mathrm{F}_{Y^*|X}(y \mid X_i) - \mathbb{E}[\mathrm{F}_{Y^*|X}(y \mid X_i)]\big)\Big\} \\
&\quad = -\frac{1}{f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))}\frac{1}{n}\sum_{i=1}^{n}\Big\{\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) - \mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big)\Big\} \\
&\qquad - \frac{1}{f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) - \mathbb{E}[\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)]\big) + o_p(1).
\end{aligned}$$

In addition to Assumption 7 and Corollary 1,

$$\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) - \mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big) = \mathbb{M}_{FY}^{L}\big(\mathrm{F}_{Y^*}^{-1}(r), X_i\big)'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}_{Y^*}^{L} - \boldsymbol{\beta}_{Y^*}^{L} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*}\end{bmatrix} + o_p(1)$$

whence

$$\frac{1}{n}\sum_{i=1}^{n}\Big\{\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) - \mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big)\Big\} = \mathbb{E}\big[\mathbb{M}_{FY}^{L}\big(\mathrm{F}_{Y^*}^{-1}(r), X\big)\big]'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}_{Y^*}^{L} - \boldsymbol{\beta}_{Y^*}^{L} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*}\end{bmatrix} + o_p(1)$$

thanks to the Weak Law of Large Numbers (WLLN), the CMT and Theorem 2. From the foregoing,

$$\begin{aligned}
&\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) - F_{Y^*}^{-1}(r)\big) \\
&\quad = -\frac{1}{f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))}\mathbb{E}\big[\mathbb{M}_{FY}^{L}\big(\mathrm{F}_{Y^*}^{-1}(r), X\big)\big]'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}_{Y^*}^{L} - \boldsymbol{\beta}_{Y^*}^{L} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*}\end{bmatrix} \\
&\qquad - \frac{1}{f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) - \mathbb{E}[\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)]\big) + o_p(1).
\end{aligned}$$

□

## SA.2 Proof of Theorem 4

The proofs of parts (b) through (f) of Theorem 4 are organized in the following lemmas.

### SA.2.1 Proof of Theorem 4(b)

The following lemma provides a proof of part (b) of Theorem 4 from the main text.

**Lemma S2.** *Suppose Assumptions 1 to 10 hold, then*

$$\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|T^*X}(y|t,x) - \mathrm{F}_{Y^*|T^*X}(y|t,x)\big) \xrightarrow{d} \mathcal{N}\big(0,\, \sigma(\mathrm{F}_{Y^*|T^*X}(y|t,x))\big).$$

*Proof.* Recall $\mathrm{F}_{Y^*|T^*X}(y|t,x) = C_{2|X}\big(\mathrm{F}_{Y^*|X}(y \mid x), \mathrm{F}_{T^*|X}(t \mid x)\big) =: C_2\big(\mathrm{F}_{Y^*|X}(y \mid x), \mathrm{F}_{T^*|X}(t \mid x);\, \delta\big)$ with $C_{2|X}(r,s) = \frac{\partial C_{Y^*T^*|X}(r,s)}{\partial s}$. The estimator is given by $\widehat{\mathrm{F}}_{Y^*|T^*X}(y|t,x) = C_2\big(\widehat{\mathrm{F}}_{Y^*|X}(y \mid x), \widehat{\mathrm{F}}_{T^*|X}(t \mid x);\, \widehat{\delta}\big)$ whence

$$\begin{aligned}
&\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|T^*X}(y|t,x) - \mathrm{F}_{Y^*|T^*X}(y|t,x)\big) \\
&\qquad = \sqrt{n}\Big(C_2\big(\widehat{\mathrm{F}}_{Y^*|X}(y \mid x), \widehat{\mathrm{F}}_{T^*|X}(t \mid x);\, \widehat{\delta}\big) - C_2\big(\mathrm{F}_{Y^*|X}(y \mid x), \mathrm{F}_{T^*|X}(t \mid x);\, \delta\big)\Big) \\
&\qquad =: \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_i^{(b)} + o_p(1)
\end{aligned}$$

where $\sigma(\mathrm{F}_{Y^*|T^*X}(y|t,x)) = \mathbb{E}\big[(\psi^{(b)})^2\big]$ following arguments analogous to that of Lemma 2. $\square$

### SA.2.2 Proof of Theorem 4(c)

The following lemma provides a proof of part (c) of Theorem 4 from the main text.

**Lemma S3.** *Suppose Assumptions 1 to 10 hold, then* $\sqrt{n}\big(\widehat{Q}_{Y^*|T^*X}(\tau \mid t,x) - Q_{Y^*|T^*X}(\tau \mid t,x)\big) \xrightarrow{d} \mathcal{N}\big(0,\, \sigma(Q_{Y^*|T^*X}(\tau \mid t,x))\big)$.

*Proof.* Recall $Q_{Y^*|T^*X}(\tau|t,x) = Q_{Y^*|X}\Big(C^{-1}_{2;1|X}\big(\tau; \mathrm{F}_{T^*|X}(t \mid x) \mid x\big)\Big) = x'\beta_{Y^*}\Big(C^{-1}_{2;1|X}\big(\tau,\, \mathrm{F}_{T^*|X}(t \mid x);\, \delta\big)\Big)$ under Assumption 2 where $C^{-1}_{2;1|X}(\cdot;\cdot)$ is the inverse of $C_{2|X}$ with respect to its first argument. The estimator is given by $\widehat{Q}_{Y^*|T^*X}(\tau|t,x) = x'\widehat{\beta}_{Y^*}\left(C^{-1}_{2;1|X}\big(\tau,\, \widehat{\mathrm{F}}_{T^*|X}(t \mid x);\, \widehat{\delta}\big)\right)$. Consider the following decomposition

$$\begin{aligned}
&\sqrt{n}\big(\widehat{Q}_{Y^*|T^*X}(\tau|t,x) - Q_{Y^*|T^*X}(\tau|t,x)\big) \\
&\qquad = \sqrt{n}x'\Big(\widehat{\beta}_{Y^*}\big(C^{-1}_{2;1|X}\big(\tau; \widehat{\mathrm{F}}_{T^*|X}(t \mid x);\, \widehat{\delta}\big)\big) - \beta_{Y^*}\big(C^{-1}_{2;1|X}\big(\tau; \mathrm{F}_{T^*|X}(t \mid x);\, \delta\big)\big)\Big) \\
&\qquad = \sqrt{n}x'\big(\widehat{\beta}_{Y^*}(\widehat{\tau}_Q) - \widehat{\beta}_{Y^*}(\tau_Q)\big) + \sqrt{n}x'\big(\widehat{\beta}_{Y^*}(\tau_Q) - \beta_{Y^*}(\tau_Q)\big) \\
&\qquad = x'\widehat{\beta}^{\partial}_{Y^*}(\bar{\tau}_Q)\sqrt{n}(\widehat{\tau}_Q - \tau_Q) + \sqrt{n}x'\big(\widehat{\beta}_{Y^*}(\tau_Q) - \beta_{Y^*}(\tau_Q)\big) \qquad \text{(S1)}
\end{aligned}$$

where $\tau_Q := C^{-1}_{2;1|X}\big(\tau;\, \tau_t\big)$, $\tau_t := \mathrm{F}_{T^*|X}(t \mid x)$, and notice that $C_{2|X}(\tau_Q,\, \tau_t;\, \delta) = \tau = C_{2|X}(\widehat{\tau}_Q,\, \widehat{\tau}_t;\, \widehat{\delta})$. Thanks to the MVT,

$$
\begin{aligned}
0 &= C_{2|X}(\widehat{\tau}_Q,\, \widehat{\tau}_t;\, \widehat{\delta}) - C_{2|X}(\tau_Q,\, \tau_t;\, \delta) \\
&= \big(C_{2|X}(\widehat{\tau}_Q,\, \widehat{\tau}_t;\, \widehat{\delta}) - C_{2|X}(\tau_Q,\, \widehat{\tau}_t;\, \widehat{\delta})\big) + \big(C_{2|X}(\tau_Q,\, \widehat{\tau}_t;\, \widehat{\delta}) - C_{2|X}(\tau_Q,\, \tau_t;\, \widehat{\delta})\big) \\
&+ \big(C_{2|X}(\tau_Q,\, \tau_t;\, \widehat{\delta}) - C_{2|X}(\tau_Q,\, \tau_t;\, \delta)\big) \\
&= \partial_1 C_{2|X}(\bar{\tau}_Q,\, \widehat{\tau}_t;\, \widehat{\delta})(\widehat{\tau}_Q - \tau_Q) + \partial_2 C_{2|X}(\tau_Q,\, \bar{\tau}_t;\, \widehat{\delta})(\widehat{\tau}_t - \tau_t) + \partial_\delta C_{2|X}(\tau_Q,\, \tau_t;\, \bar{\delta})(\widehat{\delta} - \delta) \\
&= c_{Y^*T^*|X}(\bar{\tau}_Q,\, \widehat{\tau}_t;\, \widehat{\delta})(\widehat{\tau}_Q - \tau_Q) + \partial_2 C_{2|X}(\tau_Q,\, \bar{\tau}_t;\, \widehat{\delta})(\widehat{\tau}_t - \tau_t) + \partial_\delta C_{2|X}(\tau_Q,\, \tau_t;\, \bar{\delta})(\widehat{\delta} - \delta)
\end{aligned}
$$

by the definition of the conditional copula density, i.e., $c_{Y^*T^*|X}(r,\, s;\, \cdot) = \partial_1(\partial_2 C_{Y^*T^*|X}(r,\, s;\, \cdot)) = \partial_1(C_{2|X}(r,\, s;\, \cdot))$.

From the foregoing and that the conditional copula is strictly positive (Assumption 6), apply the CMT and Theorem 2 to obtain

$$
\begin{aligned}
\sqrt{n}(\widehat{\tau}_Q - \tau_Q) &= -\big(c_{Y^*T^*|X}(\bar{\tau}_Q,\, \widehat{\tau}_t;\, \widehat{\delta})\big)^{-1}\Big(\partial_2 C_{2|X}(\tau_Q,\, \bar{\tau}_t;\, \widehat{\delta})\sqrt{n}(\widehat{\tau}_t - \tau_t) + \partial_\delta C_{2|X}(\tau_Q,\, \tau_t;\, \bar{\delta})'\sqrt{n}(\widehat{\delta} - \delta)\Big) \\
&= -\big(c_{Y^*T^*|X}(\tau_Q,\, \tau_t;\, \delta)\big)^{-1}\partial_2 C_{2|X}(\tau_Q,\, \tau_t;\, \delta)\sqrt{n}(\widehat{\mathrm{F}}_{T^*|X}(t \mid x) - \mathrm{F}_{T^*|X}(t \mid x)) \\
&\quad - \big(c_{Y^*T^*|X}(\tau_Q,\, \tau_t;\, \delta)\big)^{-1}\partial_\delta C_{2|X}(\tau_Q,\, \tau_t;\, \delta)'\sqrt{n}(\widehat{\delta} - \delta) + o_p(1).
\end{aligned}
$$

Substituting terms into (S1) using the representation in (A10), it follows from Theorem 2 and the CMT that

$$
\begin{aligned}
&\sqrt{n}\big(\widehat{Q}_{Y^*|T^*X}(\tau|t,x) - Q_{Y^*|T^*X}(\tau|t,x)\big) \\
&= \frac{1}{f_{Y^*|X}\big(C^{-1}_{2;1|X}(\tau;\, \mathrm{F}_{T^*|X}(t|x)) \mid x\big)}\sqrt{n}(\widehat{\tau}_Q - \tau_Q) \\
&\quad + \sum_{\ell=1}^{L}\omega_{\ell,\beta}(\tau_Q)\, x'\sqrt{n}\big(\widehat{\beta}_{Y^*}(\tau_\ell) - \beta_{Y^*}(\tau_\ell)\big) + o_p(1) \\
&= \underbrace{-\frac{\big(c_{Y^*T^*|X}(\tau_Q,\, \tau_t;\, \delta)\big)^{-1}}{f_{Y^*|X}\big(C^{-1}_{2;1|X}(\tau;\, \mathrm{F}_{T^*|X}(t|x)) \mid x\big)}\partial_2 C_{2|X}(\tau_Q,\, \tau_t;\, \delta)}_{\mathbb{M}^L_{Q,F}(\tau,t,x)}\sqrt{n}(\widehat{\mathrm{F}}_{T^*|X}(t \mid x) - \mathrm{F}_{T^*|X}(t \mid x)) \\
&\quad \underbrace{-\frac{\big(c_{Y^*T^*|X}(\tau_Q,\, \tau_t;\, \delta)\big)^{-1}}{f_{Y^*|X}\big(C^{-1}_{2;1|X}(\tau;\, \mathrm{F}_{T^*|X}(t|x)) \mid x\big)}\partial_\delta C_{2|X}(\tau_Q,\, \tau_t;\, \delta)'}_{\mathbb{M}^L_{Q,\delta}(\tau,t,x)'}\sqrt{n}(\widehat{\delta} - \delta) \\
&\quad + \underbrace{\Big[\omega_{1,\beta}(\tau_Q)x', \ldots, \omega_{L,\beta}(\tau_Q)x',\, 0\Big]}_{\mathbb{M}^L_{Q,\beta}(\tau,t,x)'}\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}^L_{Y^*} - \boldsymbol{\beta}^L_{Y^*} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*}\end{bmatrix} + o_p(1) \\
&= \mathbb{M}^L_{Q,F}(\tau,t,x)\sqrt{n}(\widehat{\mathrm{F}}_{T^*|X}(t \mid x) - \mathrm{F}_{T^*|X}(t \mid x)) + \mathbb{M}^L_{Q,\delta}(\tau,t,x)'\sqrt{n}(\widehat{\delta} - \delta) \\
&\quad + \mathbb{M}^L_{Q,\beta}(\tau,t,x)'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}^L_{Y^*} - \boldsymbol{\beta}^L_{Y^*} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*}\end{bmatrix} + o_p(1) \\
&=: \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_i^{(c)} + o_p(1) \xrightarrow{d} \mathcal{N}\big(0, \sigma(Q_{Y^*|T^*X}(\tau|t,x))\big)
\end{aligned}
$$

where $\sigma(Q_{Y^*|T^*X}(\tau|t,x)) = \mathbb{E}\big[(\psi^{(c)})^2\big]$. Since a linear combination of asymptotically linear quantities is asymptotically linear, the conclusion follows from Corollary 1 (applied to $T$ analogously), Proposition 2, Theorem 3, and Slutsky's Theorem. □

### SA.2.3 Proof of Theorem 4(d)

The following lemma provides a proof of part (d) of Theorem 4 from the main text.

**Lemma S4.** *Suppose Assumptions 1 to 10 hold, then*

$$\sqrt{n}\big(\widehat{\theta}_{TM}(r_1,r_2,s_1,s_2) - \theta_{TM}(r_1,r_2,s_1,s_2)\big) \xrightarrow{d} \mathcal{N}\big(0,\, \sigma(\theta_{TM}(r_1,r_2,s_1,s_2))\big).$$

*Proof.* The unconditional joint CDF can be obtained from the conditional copula:

$$\begin{aligned} F_{Y^*T^*}(y,\,t) &= \int_{\mathcal{X}} \mathrm{F}_{Y^*T^*|X}(y,\,t \mid x) d\mathrm{F}_X(x) \\ &= \int_{\mathcal{X}} C_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y \mid x), \mathrm{F}_{T^*|X}(t \mid x);\, \delta\big)\, d\mathrm{F}_X(x). \end{aligned}$$

Define, for $r = \mathrm{F}_{Y^*}(y)$ and $s = \mathrm{F}_{T^*}(t)$, the induced unconditional copula

$$\begin{aligned} C_{Y^*T^*}(r,s) &= \mathrm{F}_{Y^*T^*}\big(\mathrm{F}_{Y^*}^{-1}(r),\, \mathrm{F}_{T^*}^{-1}(s)\big) \\ &= \int_{\mathcal{X}} C_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid x), \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid x);\, \delta\big)\, d\mathrm{F}_X(x) \\ &= \mathbb{E}\Big[C_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X), \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X);\, \delta\big)\Big]. \end{aligned}$$

Recall $\theta_{TM}(r_1,r_2,s_1,s_2) = \dfrac{C_{Y^*T^*}(r_2,s_2) - C_{Y^*T^*}(r_1,s_2) - C_{Y^*T^*}(r_2,s_1) + C_{Y^*T^*}(r_1,s_1)}{s_2 - s_1}$, and the estimator is given by

$$\widehat{\theta}_{TM}(r_1,r_2,s_1,s_2) = \frac{\widehat{C}_{Y^*T^*}(r_2,s_2;\,\widehat{\delta}) - \widehat{C}_{Y^*T^*}(r_1,s_2;\,\widehat{\delta}) - \widehat{C}_{Y^*T^*}(r_2,s_1;\,\widehat{\delta}) + \widehat{C}_{Y^*T^*}(r_1,s_1;\,\widehat{\delta})}{s_2 - s_1}$$

where

$$\widehat{C}_{Y^*T^*}(r,s) := \frac{1}{n}\sum_{i=1}^{n} C_{Y^*T^*|X}\big(\widehat{\mathrm{F}}_{Y^*|X}(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) \mid X_i), \widehat{\mathrm{F}}_{T^*|X}(\widehat{\mathrm{F}}_{T^*}^{-1}(s) \mid X_i);\, \widehat{\delta}\big).$$

Let $y := \mathrm{F}_{Y^*}^{-1}(r)$, $\hat{y} := \widehat{\mathrm{F}}_{Y^*}^{-1}(r)$, $t := \mathrm{F}_{T^*}^{-1}(s)$, $\hat{t} := \widehat{\mathrm{F}}_{T^*}^{-1}(s)$, $u_i := \mathrm{F}_{Y^*|X}(y \mid X_i)$, $\hat{u}_i := \widehat{\mathrm{F}}_{Y^*|X}(\hat{y} \mid X_i)$, $v_i := \mathrm{F}_{T^*|X}(t \mid X_i)$, and $\hat{v}_i := \widehat{\mathrm{F}}_{T^*|X}(\hat{t} \mid X_i)$. Also, $C(u,v;\delta) := C_{Y^*T^*|X}(u,v;\delta)$ and its partials are $C_1 = \partial C/\partial u$, $C_2 = \partial C/\partial v$, $C_\delta = \partial C/\partial \delta$. Then, by Assumption 6 and the MVT, $C(\hat{u}_i,\hat{v}_i;\hat{\delta}) - C(u_i,v_i;\delta) = \underbrace{C_1(\tilde{u}_i,\hat{v}_i;\hat{\delta})}_{\omega_{FY}(X_i;r,s)}(\hat{u}_i - u_i) + \underbrace{C_2(u_i,\tilde{v}_i;\hat{\delta})}_{\omega_{FT}(X_i;r,s)}(\hat{v}_i - v_i) + \underbrace{C_\delta(u_i,v_i;\tilde{\delta}_i)}_{\omega_\delta(X_i;r,s)}{}'(\hat{\delta} - \delta)$.

Thanks to the MVT and the decomposition in Lemma 2, the following representation holds:

$$\begin{aligned} \sqrt{n}\big(\widehat{C}_{Y^*T^*}(r,s;\widehat{\delta}) - C_{Y^*T^*}(r,s;\delta)\big) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big\{ & C_{Y^*T^*|X}\big(\widehat{\mathrm{F}}_{Y^*|X}(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) \mid X_i), \widehat{\mathrm{F}}_{T^*|X}(\widehat{\mathrm{F}}_{T^*}^{-1}(s) \mid X_i);\, \widehat{\delta}\big) \\ & - \mathbb{E}\Big[C_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X), \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X);\, \delta\big)\Big]\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n} \omega_{FY}(X_i; r, s)\,\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|X} - \mathrm{F}_{Y^*|X}\big)(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n} \omega_{FT}(X_i; r, s)\,\sqrt{n}\big(\widehat{\mathrm{F}}_{T^*|X} - \mathrm{F}_{T^*|X}\big)(\mathrm{F}_{T^*}^{-1}(s) \mid X_i) \\
&\quad + \Big(\frac{1}{n}\sum_{i=1}^{n} \omega_{QY}(X_i; r, s)\Big)\,\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) - \mathrm{F}_{Y^*}^{-1}(r)\big) \\
&\quad + \Big(\frac{1}{n}\sum_{i=1}^{n} \omega_{QT}(X_i; r, s)\Big)\,\sqrt{n}\big(\widehat{\mathrm{F}}_{T^*}^{-1}(s) - \mathrm{F}_{T^*}^{-1}(s)\big) \\
&\quad + \Big(\frac{1}{n}\sum_{i=1}^{n} \omega_{\delta}(X_i; r, s)\Big)'\,\sqrt{n}(\hat{\delta} - \delta) \\
&\quad + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \Big\{ C(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i); \delta) \\
&\qquad\qquad - \mathbb{E}\big[C(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i); \delta)\big] \Big\}.
\end{aligned}
$$

where

$$
\omega_{QY}(X_i; r, s) := \omega_{FY}(X_i; r, s)\,\widehat{f}_{Y^*|X}(\bar{y}_i \mid X_i), \text{ and } \qquad \omega_{QT}(X_i; r, s) := \omega_{FT}(X_i; r, s)\,\widehat{f}_{T^*|X}(\bar{t}_i \mid X_i).
$$

First, by Assumption 7 and Corollary 1(a),

$$
\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|X} - \mathrm{F}_{Y^*|X}\big)(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) = \mathbb{M}_{FY}^{L}(\mathrm{F}_{Y^*}^{-1}(r), X_i)'\sqrt{n}\begin{bmatrix} \widehat{\boldsymbol{\beta}}_{Y^*}^{L} - \boldsymbol{\beta}_{Y^*}^{L} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*} \end{bmatrix} + o_p(1).
$$

Second, by Lemma S1 and Assumption 7,

$$
\begin{aligned}
&\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) - F_{Y^*}^{-1}(r)\big) \\
&\quad = -\frac{1}{f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))}\mathbb{E}\big[\mathbb{M}_{FY}^{L}\big(\mathrm{F}_{Y^*}^{-1}(r), X\big)\big]'\sqrt{n}\begin{bmatrix} \widehat{\boldsymbol{\beta}}_{Y^*}^{L} - \boldsymbol{\beta}_{Y^*}^{L} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*} \end{bmatrix} \\
&\qquad - \frac{1}{f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) - \mathbb{E}[\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)]\big) + o_p(1).
\end{aligned}
$$

Third, by Proposition 2,

$$
\sqrt{n}(\widehat{\delta} - \delta) = -\mathcal{H}_{\delta}^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathcal{S}_i(\delta) - \mathcal{H}_{\delta}^{-1}\mathbb{M}_{\Delta Y}^{L}\sqrt{n}\begin{bmatrix} \widehat{\boldsymbol{\beta}}_{Y^*}^{L} - \boldsymbol{\beta}_{Y^*}^{L} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*} \end{bmatrix} - \mathcal{H}_{\delta}^{-1}\mathbb{M}_{\Delta T}^{L}\sqrt{n}\begin{bmatrix} \widehat{\boldsymbol{\beta}}_{T^*}^{L} - \boldsymbol{\beta}_{T^*}^{L} \\ \widehat{\sigma}_{T^*} - \sigma_{T^*} \end{bmatrix} + o_p(1)
$$

Lastly, the summands

$$
\Big\{C(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i); \delta) - \mathbb{E}\big[C(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i); \delta)\big]\Big\}_{i=1}^{n}
$$

are *i.i.d*, mean-zero, with bounded second moments uniformly in $(r, s, \delta')' \in (0,\,1)^2 \times \Gamma_{\delta}$ by the definition of a copula function.

In sum, $\sqrt{n}\big(\widehat{C}_{Y^*T^*}(r, s; \widehat{\delta}) - C_{Y^*T^*}(r, s; \delta)\big)$ can be expressed as a weighted sum of asymptotically normal quantities up to a $o_p(1)$ term:

$$\begin{aligned}
&\sqrt{n}\big(\widehat{C}_{Y^*T^*}(r,s;\widehat{\delta}) - C_{Y^*T^*}(r,s;\delta)\big) \\
&\quad = \mathbb{M}^L_{CY}(r,s)'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}^L_{Y^*} - \boldsymbol{\beta}^L_{Y^*} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*}\end{bmatrix} \;+\; \mathbb{M}^L_{CT}(r,s)'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}^L_{T^*} - \boldsymbol{\beta}^L_{T^*} \\ \widehat{\sigma}_{T^*} - \sigma_{T^*}\end{bmatrix} \\
&\qquad + \mathbb{M}^L_{CF^{-1}_{Y^*}}(r,s)\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(\mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r)\mid X_i) - \mathbb{E}[\mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r)\mid X_i)]\big) \\
&\qquad + \mathbb{M}^L_{CF^{-1}_{T^*}}(r,s)\frac{1}{\sqrt{n}}\sum_{i=1}^n \big(\mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s)\mid X_i) - \mathbb{E}[\mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s)\mid X_i)]\big) \\
&\qquad + \frac{1}{\sqrt{n}}\sum_{i=1}^n \Big\{C(\mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r)\mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s)\mid X_i);\delta) \\
&\qquad\qquad - \mathbb{E}\big[C(\mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r)\mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s)\mid X_i);\delta)\big]\Big\} \\
&\qquad + \mathbb{M}^L_{C\delta}(r,s)'\frac{1}{\sqrt{n}}\sum_{i=1}^n \mathcal{S}_i(\delta) + o_p(1) \\
&\quad =: \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi_i^{(d)}(r,s) + o_p(1)
\end{aligned} \tag{S2}$$

where the representation in the last line follows because the linear combination of asymptotically linear quantities is asymptotically linear. The conclusion then follows in addition Theorem 3, Proposition 2, and Slutsky's Theorem noting that

$$\begin{aligned}
&\sqrt{n}\big(\widehat{\theta}_{TM}(r_1,r_2,s_1,s_2) - \theta_{TM}(r_1,r_2,s_1,s_2)\big) = (s_2-s_1)^{-1}\sqrt{n}\big(\widehat{C}_{Y^*T^*}(r_2,s_2;\widehat{\delta}) - C_{Y^*T^*}(r_2,s_2;\delta)\big) \\
&\; -(s_2-s_1)^{-1}\sqrt{n}\big(\widehat{C}_{Y^*T^*}(r_1,s_2;\widehat{\delta}) - C_{Y^*T^*}(r_1,s_2;\delta)\big) - (s_2-s_1)^{-1}\sqrt{n}\big(\widehat{C}_{Y^*T^*}(r_2,s_1;\widehat{\delta}) - C_{Y^*T^*}(r_2,s_1;\delta)\big) \\
&\; +(s_2-s_1)^{-1}\sqrt{n}\big(\widehat{C}_{Y^*T^*}(r_1,s_1;\widehat{\delta}) - C_{Y^*T^*}(r_1,s_1;\delta)\big) \\
&\quad =: (s_2-s_1)^{-1}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n \psi_i^{(d)}(r_2,s_2) - \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi_i^{(d)}(r_1,s_2) - \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi_i^{(d)}(r_2,s_1) + \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi_i^{(d)}(r_1,s_1)\Big) + o_p(1) \\
&\quad =: \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi_i^{(d)} + o_p(1)
\end{aligned}$$

where $\sigma(\theta_{TM}(r_1,r_2,s_1,s_2)) = \mathbb{E}\big[(\psi^{(d)})^2\big]$. □

### SA.2.4 Proof of Theorem 4(e)

The following lemma provides a proof of part (e) of Theorem 4 from the main text.

**Lemma S5.** *Under Assumptions 1 to 10,* $\sqrt{n}(\widehat{\rho}_S - \rho_S) \xrightarrow{d} \mathcal{N}\big(0,\, \sigma(\rho_S)\big)$.

*Proof.* Recall $\rho_S = 12\int_0^1\int_0^1 C_{Y^*T^*}(r,s)\,dr\,ds - 3$ where

$$C_{Y^*T^*}(r,s) = \mathbb{E}\Big[C_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r)\mid X),\; \mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s)\mid X);\,\delta\big)\Big].$$

From (S2) in the proof of Lemma S4,

$$
\begin{aligned}
\sqrt{n}(\widehat{\rho}_S - \rho_S)/12 &= \sqrt{n} \int_0^1 \int_0^1 \big(\widehat{C}_{Y^*T^*}(r,s) - C_{Y^*T^*}(r,s)\big)\, dr\, ds \\
&= \Big(\int_0^1 \int_0^1 \mathbb{M}^L_{CY}(r,s)\, dr\, ds\Big)' \sqrt{n} \begin{bmatrix} \widehat{\boldsymbol{\beta}}^L_{Y^*} - \boldsymbol{\beta}^L_{Y^*} \\ \widehat{\sigma}_{Y^*} - \sigma_{Y^*} \end{bmatrix} \;+\; \Big(\int_0^1 \int_0^1 \mathbb{M}^L_{CT}(r,s)\, dr\, ds\Big)' \sqrt{n} \begin{bmatrix} \widehat{\boldsymbol{\beta}}^L_{T^*} - \boldsymbol{\beta}^L_{T^*} \\ \widehat{\sigma}_{T^*} - \sigma_{T^*} \end{bmatrix} \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big\{ \Big( \int_0^1 \int_0^1 \mathbb{M}^L_{CF^{-1}_{Y^*}}(r,s) \mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r) \mid X_i)\, dr\, ds \Big) \\
&\qquad\qquad - \mathbb{E}\Big[ \int_0^1 \int_0^1 \mathbb{M}^L_{CF^{-1}_{Y^*}}(r,s) \mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r) \mid X_i)\, dr\, ds \Big] \Big\} \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big\{ \Big( \int_0^1 \int_0^1 \mathbb{M}^L_{CF^{-1}_{T^*}}(r,s) \mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s) \mid X_i)\, dr\, ds \Big) \\
&\qquad\qquad - \mathbb{E}\Big[ \int_0^1 \int_0^1 \mathbb{M}^L_{CF^{-1}_{T^*}}(r,s) \mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s) \mid X_i)\, dr\, ds \Big] \Big\} \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big\{ \int_0^1 \int_0^1 C(\mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r) \mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s) \mid X_i); \delta)\, dr\, ds \\
&\qquad\qquad - \mathbb{E}\Big[ \int_0^1 \int_0^1 C(\mathrm{F}_{Y^*|X}(\mathrm{F}^{-1}_{Y^*}(r) \mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}^{-1}_{T^*}(s) \mid X_i); \delta)\, dr\, ds \Big] \Big\} \\
&\quad + \Big(\int_0^1 \int_0^1 \mathbb{M}^L_{C\delta}(r,s)\, dr\, ds\Big)' \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathcal{S}_i(\delta) + o_p(1) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big\{ \int_0^1 \int_0^1 \psi_i^{(d)}(r,s)\, dr\, ds \Big\} + o_p(1) \\
&=: \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_i^{(e)} + o_p(1) \xrightarrow{d} \mathcal{N}(0, \sigma(\rho_S))
\end{aligned}
$$

where $\sigma(\rho_S) = \mathbb{E}\big[(\psi_i^{(e)})^2\big]$. The conclusion follows from Lemma S4. □

### SA.2.5 Proof of Theorem 4(f)

The following lemma provides a proof of part (f) of Theorem 4 from the main text.

**Lemma S6.** *Suppose Assumptions 1 to 10 hold, then* $\sqrt{n}\big(\widehat{\theta}_U(\Delta, s_1, s_2) - \theta_U(\Delta, s_1, s_2)\big) \xrightarrow{d} \mathcal{N}\big(0, \sigma(\theta_U(\Delta, s_1, s_2))\big)$.

*Proof.* Recall $\theta_U(\Delta, s_1, s_2) := \dfrac{\mathrm{P}\big(\mathrm{F}_{Y^*}(Y^*) > \mathrm{F}_{T^*}(T^*) + \Delta,\; s_1 \le \mathrm{F}_{T^*}(T^*) \le s_2\big)}{s_2 - s_1}$. A first step is to obtain the joint unconditional pdf $f_{Y^*T^*}(\cdot, \cdot)$ from the conditional copula density:

$$
\begin{aligned}
f_{Y^*T^*}(y, t) &= \int_{\mathcal{X}} f_{Y^*T^*|X}(y, t \mid x) d\mathrm{F}_X(x) \\
&= \int_{\mathcal{X}} c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y \mid x), \mathrm{F}_{T^*|X}(t \mid x); \delta\big) f_{Y^*|X}(y \mid x) f_{T^*|X}(t \mid x) d\mathrm{F}_X(x).
\end{aligned}
$$

Applying a change of variables $r = \mathrm{F}_{Y^*}(y)$ and $s = \mathrm{F}_{T^*}(t)$, then $y = \mathrm{F}^{-1}_{Y^*}(r)$, $t = \mathrm{F}^{-1}_{T^*}(s)$, $dy = dr/f_{Y^*}(y) = dr/f_{Y^*}(\mathrm{F}^{-1}_{Y^*}(r))$, and $dt = ds/f_{T^*}(t) = ds/f_{T^*}(\mathrm{F}^{-1}_{T^*}(s))$. Thus

$$
\begin{aligned}
&\mathrm{P}\big(\mathrm{F}_{Y^*}(Y^*) > \mathrm{F}_{T^*}(T^*) + \Delta,\ s_1 \le \mathrm{F}_{T^*}(T^*) \le s_2\big)\\
&\qquad = \iint \mathbf{1}\{\mathrm{F}_{Y^*}(y) > \mathrm{F}_{T^*}(t) + \Delta\}\, \mathbf{1}\{s_1 \le \mathrm{F}_{T^*}(t) \le s_2\}\, f_{Y^*T^*}(y,t)\, dy\, dt\\
&\qquad = \int_0^1\!\!\int_0^1 \mathbf{1}\{r > s + \Delta\}\ \mathbf{1}\{s_1 \le s \le s_2\}\ \frac{f_{Y^*T^*}\big(\mathrm{F}_{Y^*}^{-1}(r),\ \mathrm{F}_{T^*}^{-1}(s)\big)}{f_{Y^*}\big(\mathrm{F}_{Y^*}^{-1}(r)\big)\ f_{T^*}\big(\mathrm{F}_{T^*}^{-1}(s)\big)}\, dr\, ds.
\end{aligned}
$$

The plug-in estimator via numerical integration is given by

$$
\widehat{\theta}_U(\Delta, s_1, s_2) := \int_0^1\!\!\int_0^1 \frac{\mathbf{1}\{r > s + \Delta\}\ \mathbf{1}\{s_1 \le s \le s_2\}}{(s_2 - s_1)}\ \frac{\widehat{f}_{Y^*T^*}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r),\ \widehat{\mathrm{F}}_{T^*}^{-1}(s)\big)}{\widehat{f}_{Y^*}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r)\big)\ \widehat{f}_{T^*}\big(\widehat{\mathrm{F}}_{T^*}^{-1}(s)\big)}\, dr\, ds.
$$

Consider the decomposition:

$$
\begin{aligned}
&\frac{\widehat{f}_{Y^*T^*}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r),\ \widehat{\mathrm{F}}_{T^*}^{-1}(s)\big)}{\widehat{f}_{Y^*}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r)\big)\ \widehat{f}_{T^*}\big(\widehat{\mathrm{F}}_{T^*}^{-1}(s)\big)} - \frac{f_{Y^*T^*}\big(\mathrm{F}_{Y^*}^{-1}(r),\ \mathrm{F}_{T^*}^{-1}(s)\big)}{f_{Y^*}\big(\mathrm{F}_{Y^*}^{-1}(r)\big)\ f_{T^*}\big(\mathrm{F}_{T^*}^{-1}(s)\big)} = \frac{\widehat{f}_{Y^*T^*}(y,t) - f_{Y^*T^*}(y,t)}{f_{Y^*}(y) f_{T^*}(t)}\\
&\quad + \frac{\partial_y \widehat{f}_{Y^*T^*}(\bar{y}_1, \bar{t}_1)\,(\widehat{y} - y) + \partial_t \widehat{f}_{Y^*T^*}(\bar{y}_2, \bar{t}_2)\,(\widehat{t} - t)}{f_{Y^*}(y) f_{T^*}(t)}\\
&\quad + \frac{\widehat{f}_{Y^*T^*}(\widehat{y}, \widehat{t})}{f_{Y^*}(y) f_{T^*}(t)\, \widehat{f}_{Y^*}(\widehat{y}) \widehat{f}_{T^*}(\widehat{t})}\\
&\qquad \times \Big\{ \big[f_{Y^*}(y) - \widehat{f}_{Y^*}(y)\big]\, f_{T^*}(t) + f_{Y^*}(y)\, \big[f_{T^*}(t) - \widehat{f}_{T^*}(t)\big] - \widehat{f}_{T^*}(\widehat{t})\, \widehat{f}'_{Y^*}(\bar{y})\,(\widehat{y} - y) - \widehat{f}_{Y^*}(y)\, \widehat{f}'_{T^*}(\bar{t})\,(\widehat{t} - t) \Big\}\\
&= \omega_f(y,t)\,\big(\widehat{f}_{Y^*T^*}(y,t) - f_{Y^*T^*}(y,t)\big) \ +\ \widehat{\omega}_{fy}(\widehat{y}, \widehat{t}, y)\,\big(\widehat{f}_{Y^*}(y) - f_{Y^*}(y)\big) \ +\ \widehat{\omega}_{ft}(\widehat{y}, \widehat{t}, t)\,\big(\widehat{f}_{T^*}(t) - f_{T^*}(t)\big)\\
&\qquad +\ \widehat{\omega}_y(\widehat{y}, \widehat{t}, y, t)\,(\widehat{y} - y) \ +\ \widehat{\omega}_t(\widehat{y}, \widehat{t}, y, t)\,(\widehat{t} - t)
\end{aligned}
$$

with weights given by $\omega_f(y,t) = \frac{1}{f_{Y^*}(y) f_{T^*}(t)}$, $\widehat{\omega}_{fy}(\widehat{y}, \widehat{t}, y) = -\dfrac{\widehat{f}_{Y^*T^*}(\widehat{y}, \widehat{t})}{f_{Y^*}(y)\, \widehat{f}_{Y^*}(\widehat{y})\, \widehat{f}_{T^*}(\widehat{t})}$, $\widehat{\omega}_{ft}(\widehat{y}, \widehat{t}, t) = -\dfrac{\widehat{f}_{Y^*T^*}(\widehat{y}, \widehat{t})}{f_{T^*}(t)\, \widehat{f}_{Y^*}(\widehat{y})\, \widehat{f}_{T^*}(\widehat{t})}$, $\widehat{\omega}_y(\widehat{y}, \widehat{t}, y, t) = \dfrac{\partial_y \widehat{f}_{Y^*T^*}(\bar{y}_1, \bar{t}_1)}{f_{Y^*}(y) f_{T^*}(t)} - \dfrac{\widehat{f}_{Y^*T^*}(\widehat{y}, \widehat{t})\, \widehat{f}_{T^*}(\widehat{t})}{f_{Y^*}(y) f_{T^*}(t)\, \widehat{f}_{Y^*}(\widehat{y})\, \widehat{f}_{T^*}(\widehat{t})}\, \widehat{f}'_{Y^*}(\bar{y})$, and $\widehat{\omega}_t(\widehat{y}, \widehat{t}, y, t) = \dfrac{\partial_t \widehat{f}_{Y^*T^*}(\bar{y}_2, \bar{t}_2)}{f_{Y^*}(y) f_{T^*}(t)} - \dfrac{\widehat{f}_{Y^*T^*}(\widehat{y}, \widehat{t})\, \widehat{f}_{Y^*}(y)}{f_{Y^*}(y) f_{T^*}(t)\, \widehat{f}_{Y^*}(\widehat{y})\, \widehat{f}_{T^*}(\widehat{t})}\, \widehat{f}'_{T^*}(\bar{t})$.

Thus,

$$
\begin{aligned}
&\sqrt{n}\big(\widehat{\theta}_U(\Delta, s_1, s_2) - \theta_U(\Delta, s_1, s_2)\big)\\
&\quad = \int_0^1\!\!\int_0^1 \frac{\mathbf{1}\{r > s + \Delta\}\ \mathbf{1}\{s_1 \le s \le s_2\}}{(s_2 - s_1)} \left( \frac{\widehat{f}_{Y^*T^*}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r),\ \widehat{\mathrm{F}}_{T^*}^{-1}(s)\big)}{\widehat{f}_{Y^*}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r)\big)\ \widehat{f}_{T^*}\big(\widehat{\mathrm{F}}_{T^*}^{-1}(s)\big)} - \frac{f_{Y^*T^*}\big(\mathrm{F}_{Y^*}^{-1}(r),\ \mathrm{F}_{T^*}^{-1}(s)\big)}{f_{Y^*}\big(\mathrm{F}_{Y^*}^{-1}(r)\big)\ f_{T^*}\big(\mathrm{F}_{T^*}^{-1}(s)\big)} \right) dr\, ds\\
&\quad = \int_0^1\!\!\int_0^1 \frac{\mathbf{1}\{r > s + \Delta\}\ \mathbf{1}\{s_1 \le s \le s_2\}}{(s_2 - s_1)}\, \omega_f(y,t)\, \sqrt{n}\big(\widehat{f}_{Y^*T^*}(\mathrm{F}_{Y^*}^{-1}(r), F_{T^*}^{-1}(s)) - f_{Y^*T^*}(\mathrm{F}_{Y^*}^{-1}(r), F_{T^*}^{-1}(s))\big)\, dr\, ds\\
&\quad + \int_0^1 \left( \int_0^1 \frac{\mathbf{1}\{r > s + \Delta\}\ \mathbf{1}\{s_1 \le s \le s_2\}}{(s_2 - s_1)}\, \widehat{\omega}_{fy}(\widehat{y}, \widehat{t}, y)\, ds \right) \sqrt{n}\big(\widehat{f}_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r)) - f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))\big)\, dr\\
&\quad + \int_0^1 \left( \int_0^1 \frac{\mathbf{1}\{r > s + \Delta\}\ \mathbf{1}\{s_1 \le s \le s_2\}}{(s_2 - s_1)}\, \widehat{\omega}_{ft}(\widehat{y}, \widehat{t}, t)\, dr \right) \sqrt{n}\big(\widehat{f}_{T^*}(\mathrm{F}_{T^*}^{-1}(s)) - f_{T^*}(\mathrm{F}_{T^*}^{-1}(s))\big)\, ds\\
&\quad + \int_0^1 \left( \int_0^1 \frac{\mathbf{1}\{r > s + \Delta\}\ \mathbf{1}\{s_1 \le s \le s_2\}}{(s_2 - s_1)}\, \widehat{\omega}_y(\widehat{y}, \widehat{t}, y, t)\ ds \right) \sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*}^{-1}(r) - \mathrm{F}_{Y^*}^{-1}(r)\big)\, dr
\end{aligned}
$$

$$+\int_0^1\left(\int_0^1 \frac{\mathbf{1}\{r>s+\Delta\}\ \mathbf{1}\{s_1\le s\le s_2\}}{(s_2-s_1)}\,\widehat{\omega}_t(\widehat{y},\widehat{t},y,t)\,dr\right)\sqrt{n}\big(\widehat{\mathrm{F}}_{T^*}^{-1}(s)-\mathrm{F}_{T^*}^{-1}(s)\big)\,ds.$$

The summands are studied in turn.

First, by the MVT,

$$\begin{aligned}
&\sqrt{n}\big(\widehat{f}_{Y^*T^*}(y,t)-f_{Y^*T^*}(y,t)\big)\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big[c_{Y^*T^*|X}\big(\widehat{\mathrm{F}}_{Y^*|X}(y\mid X_i),\widehat{\mathrm{F}}_{T^*|X}(t\mid X_i);\,\widehat{\delta}\big)\widehat{f}_{Y^*|X}(y\mid X_i)\widehat{f}_{T^*|X}(t\mid X_i)\Big]\\
&\quad-\mathbb{E}\Big[c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y\mid X),\mathrm{F}_{T^*|X}(t\mid X);\,\delta\big)f_{Y^*|X}(y\mid X)f_{T^*|X}(t\mid X)\Big]\\
&=\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big\{c_{Y^*T^*|X}\big(\widehat{\mathrm{F}}_{Y^*|X}(y\mid X_i),\widehat{\mathrm{F}}_{T^*|X}(t\mid X_i);\,\widehat{\delta}\big)\widehat{f}_{Y^*|X}(y\mid X_i)\widehat{f}_{T^*|X}(t\mid X_i)\\
&\qquad\qquad-c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y\mid X_i),\mathrm{F}_{T^*|X}(t\mid X_i);\,\delta\big)f_{Y^*|X}(y\mid X_i)f_{T^*|X}(t\mid X_i)\Big\}\\
&\quad+\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big\{c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y\mid X_i),\mathrm{F}_{T^*|X}(t\mid X_i);\,\delta\big)f_{Y^*|X}(y\mid X_i)f_{T^*|X}(t\mid X_i)\\
&\qquad\qquad-\mathbb{E}\Big[c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y\mid X_i),\mathrm{F}_{T^*|X}(t\mid X_i);\,\delta\big)f_{Y^*|X}(y\mid X_i)f_{T^*|X}(t\mid X_i)\Big]\Big\}
\end{aligned}$$

$$\begin{aligned}
=\;&\frac{1}{n}\sum_{i=1}^n\omega_{FY}(X_i;y,t)\sqrt{n}\big(\widehat{\mathrm{F}}_{Y^*|X}(y\mid X_i)-\mathrm{F}_{Y^*|X}(y\mid X_i)\big)\\
&+\frac{1}{n}\sum_{i=1}^n\omega_{FT}(X_i;y,t)\sqrt{n}\big(\widehat{\mathrm{F}}_{T^*|X}(t\mid X_i)-\mathrm{F}_{T^*|X}(t\mid X_i)\big)\\
&+\frac{1}{n}\sum_{i=1}^n\omega_{fY}(X_i;y,t)\sqrt{n}\big(\widehat{f}_{Y^*|X}(y\mid X_i)-f_{Y^*|X}(y\mid X_i)\big)\\
&+\frac{1}{n}\sum_{i=1}^n\omega_{fT}(X_i;y,t)\sqrt{n}\big(\widehat{f}_{T^*|X}(t\mid X_i)-f_{T^*|X}(t\mid X_i)\big)\\
&+\frac{1}{n}\sum_{i=1}^n\omega_{\delta}(X_i;y,t)'\sqrt{n}\big(\widehat{\delta}-\delta\big)\\
&+\frac{1}{\sqrt{n}}\sum_{i=1}^n\Big\{c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y\mid X_i),\mathrm{F}_{T^*|X}(t\mid X_i);\,\delta\big)f_{Y^*|X}(y\mid X_i)f_{T^*|X}(t\mid X_i)\\
&\qquad-\mathbb{E}\Big[c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y\mid X_i),\mathrm{F}_{T^*|X}(t\mid X_i);\,\delta\big)f_{Y^*|X}(y\mid X_i)f_{T^*|X}(t\mid X_i)\Big]\Big\}
\end{aligned}$$

where the weights are given by

$$\begin{aligned}
&\omega_{FY}(X_i;y,t)=c_1(\bar{u}_i,\widehat{v}_i;\widehat{\delta})\,\widehat{f}_{Y^*|X}(y\mid X_i)\widehat{f}_{T^*|X}(t\mid X_i),\quad \omega_{FT}(X_i;y,t)=c_2(u_i,\bar{v}_i;\widehat{\delta})\,\widehat{f}_{Y^*|X}(y\mid X_i)\widehat{f}_{T^*|X}(t\mid X_i),\\
&\omega_{fY}(X_i;y,t)=c(u_i,v_i;\widehat{\delta})\,\widehat{f}_{T^*|X}(t\mid X_i),\quad \omega_{fT}(X_i;y,t)=c(u_i,v_i;\widehat{\delta})\,f_{Y^*|X}(y\mid X_i),\\
&\omega_{\delta}(X_i;y,t)=c_{\delta}(u_i,v_i;\bar{\delta}_i)\,f_{Y^*|X}(y\mid X_i)f_{T^*|X}(t\mid X_i),\ u_i:=\mathrm{F}_{Y^*|X}(y\mid X_i),\ \hat{u}_i:=\widehat{\mathrm{F}}_{Y^*|X}(y\mid X_i),\\
&\qquad v_i:=\mathrm{F}_{T^*|X}(t\mid X_i),\quad\text{and }\hat{v}_i:=\widehat{\mathrm{F}}_{T^*|X}(t\mid X_i).
\end{aligned}$$

The summands of

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big\{c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y \mid X_i),\mathrm{F}_{T^*|X}(t \mid X_i);\,\delta\big)f_{Y^*|X}(y \mid X_i)f_{T^*|X}(t \mid X_i) \\ -\mathbb{E}\Big[c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(y \mid X_i),\mathrm{F}_{T^*|X}(t \mid X_i);\,\delta\big)f_{Y^*|X}(y \mid X_i)f_{T^*|X}(t \mid X_i)\Big]\Big\}$$

are *i.i.d.* under Assumption 4, mean-zero, with bounded second moments under Assumptions 6 and 10 — the CLT applies.

Second, under the conditions of Lemma 3(a),

$$\begin{aligned}\sqrt{n}\big(\widehat{f}_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))-f_{Y^*}(\mathrm{F}_{Y^*}^{-1}(r))\big) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(\widehat{f}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)-\mathbb{E}\big[f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big]\big) \\ &= \frac{1}{n}\sum_{i=1}^{n}\sqrt{n}\big(\widehat{f}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)-f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big) \\ &+ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)-\mathbb{E}\big[f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big]\big) \\ &= \sum_{\ell=1}^{L}\Big\{\frac{1}{n}\sum_{i=1}^{n}\widehat{\mathbb{R}}_{\ell,f_{Y^*|X}}(\mathrm{F}_{Y^*}^{-1}(r),X_i)\Big\}'\sqrt{n}\big(\widehat{\beta}_{Y^*}(\tau_\ell)-\beta_{Y^*}(\tau_\ell)\big) \\ &+ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)-\mathbb{E}\big[f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big]\big) \\ &= \mathbb{E}\big[\mathbb{M}_{fY}^{L}\big(\mathrm{F}_{Y^*}^{-1}(r),X\big)\big]'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}_{Y^*}^{L}-\boldsymbol{\beta}_{Y^*}^{L}\\ \widehat{\sigma}_{Y^*}-\sigma_{Y^*}\end{bmatrix} \\ &+ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\big(f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)-\mathbb{E}\big[f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big]\big)+o_p(1)\end{aligned}$$

where

$$\mathbb{M}_{fY}^{L}(y,x) := \operatorname*{plim}_{n\to\infty}\Big[\widehat{\mathbb{R}}_{1,f_{Y^*|X}}(y,x)',\ldots,\widehat{\mathbb{R}}_{L,f_{Y^*|X}}(y,x)',\,0\Big]'.$$

Under Assumption 4 and Assumption 10(d), the summands $\big\{f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)-\mathbb{E}\big[f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\big]\big\}_{i=1}^{n}$ are *i.i.d.*, mean-zero with bounded second moments.

Putting together the foregoing, it follows from Corollary 1, lemmas S1 and 3, and proposition 2 that $\sqrt{n}\big(\widehat{\theta}_U(\Delta,s_1,s_2)-\theta_U(\Delta,s_1,s_2)\big)$ has the representation of a weighted sum of asymptotically normally linear and asymptotically normally distributed quantities:

$$\begin{aligned}&\sqrt{n}\big(\widehat{\theta}_U(\Delta,s_1,s_2)-\theta_U(\Delta,s_1,s_2)\big) \\ &\quad= \Big(\int_0^1\int_0^1 \mathbb{M}_{\theta Y}^{L}(r,s)\,dr\,ds\Big)'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}_{Y^*}^{L}-\boldsymbol{\beta}_{Y^*}^{L}\\ \widehat{\sigma}_{Y^*}-\sigma_{Y^*}\end{bmatrix} + \Big(\int_0^1\int_0^1 \mathbb{M}_{\theta T}^{L}(r,s)\,dr\,ds\Big)'\sqrt{n}\begin{bmatrix}\widehat{\boldsymbol{\beta}}_{T^*}^{L}-\boldsymbol{\beta}_{T^*}^{L}\\ \widehat{\sigma}_{T^*}-\sigma_{T^*}\end{bmatrix} \\ &\quad+ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big\{\int_0^1\int_0^1 \mathbb{M}_{\theta f}^{L}(r,s)c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i),\mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i);\,\delta\big)\end{aligned}$$

$$
\begin{aligned}
&\quad\times f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) f_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i)\, dr\, ds \\
&\quad - \mathbb{E}\Big[ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta f}(r,s) c_{Y^*T^*|X}\big(\mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i), \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i); \delta\big) \\
&\quad\times f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i) f_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i)\, dr\, ds \Big]\Big\} \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big\{ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta f_{Y^*|X}}(r,s) f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\, dr\, ds \\
&\quad - \mathbb{E}\Big[ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta f_{Y^*|X}}(r,s) f_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\, dr\, ds \Big]\Big\} \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big\{ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta f_{T^*|X}}(r,s) f_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i)\, dr\, ds \\
&\quad - \mathbb{E}\Big[ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta f_{T^*|X}}(r,s) f_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i)\, dr\, ds \Big]\Big\} \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big\{ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta F^{-1}_{Y^*}}(r,s) \mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\, dr\, ds - \mathbb{E}\Big[ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta F^{-1}_{Y^*}}(r,s) \mathrm{F}_{Y^*|X}(\mathrm{F}_{Y^*}^{-1}(r) \mid X_i)\, dr\, ds \Big]\Big\} \\
&+ \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big\{ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta F^{-1}_{T^*}}(r,s) \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i)\, dr\, ds - \mathbb{E}\Big[ \int_0^1 \int_0^1 \mathbb{M}^L_{\theta F^{-1}_{T^*}}(r,s) \mathrm{F}_{T^*|X}(\mathrm{F}_{T^*}^{-1}(s) \mid X_i)\, dr\, ds \Big]\Big\} \\
&+ \Big( \int_0^1 \int_0^1 \mathbb{M}^L_{\theta\delta}(r,s)\, dr\, ds \Big)' \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathcal{S}_i(\delta) + o_p(1) \\
&=: \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_i^{(f)} + o_p(1) \xrightarrow{d} \mathcal{N}\big(0,\, \sigma(\theta_U(\Delta, s_1, s_2))\big)
\end{aligned}
$$

where $\sigma(\theta_U(\Delta, s_1, s_2)) = \mathbb{E}\big[(\psi^{(f)})^2\big]$. □

## SB Causal Effects

Our framework is also related to the literature on continuous treatment effects (see, for example, Hirano and Imbens (2004), Flores (2007), Flores, Flores-Lagunes, Gonzalez, and Neumann (2012), and Galvao and Wang (2015)).[1] Like that literature, we consider the case with one particular continuous "treatment" variable whose effect on the outcome is of primary interest, but with other covariates that need to be controlled for. Let $Y^*(t)$ denote an individual's "potential" outcome if they experience treatment $t$; note that this is well-defined regardless of what treatment level a particular individual actually experiences, but (in the absence of measurement error) only $Y^* = Y^*(T^*)$ is observed. Next, we show that several of our parameters of interest in Section 2 correspond to the parameters of interest in the continuous treatment effect literature under the commonly invoked assumption of unconfoundedness.

**Assumption U** (Unconfoundedness)**.** *For all $t \in$ support$(T^*)$,*

$$Y^*(t) \perp\!\!\!\perp T^* | X$$

Assumption U says that, after conditioning on covariates $X$, the amount/intensity of the treatment is as

[1] See Haavelmo (1944) (in particular, the discussion about "potential influence" in Section 7) for a much earlier discussion of causality with continuous treatments that is conceptually similar to the references above.

good as randomly assigned. This is a strong assumption; it seems unlikely that it would hold in the context of intergenerational mobility — to be clear, we do not maintain this assumption in our application but note it here to show that our approach could deliver causally interpretable parameters if Assumption U holds in a given application. That being said, Assumption U is the leading assumption in the continuous treatment effects literature (as well as being a leading assumption in the binary treatment effects literature), and a similar assumption has been made in each of the continuous treatment effect papers cited above. Under Assumption U, it is straightforward to show that

$$\mathrm{P}\big(Y^*(t) \leq y | X = x\big) = \mathrm{F}_{Y^*|T^*X}(y|t, x)$$

which implies that the distribution of potential outcomes is the same as the distribution in Equation (1) in the main text. Thus, Assumption U serves to give terms like those in Equation (1) a causal interpretation as in the next example.

**Example S1** (Dose-Response Functions and Distributional Treatment Effects)**.**

The treatment effects mentioned above depend on particular values of the covariates, $x$. Often, researchers would like to report a summary measure that integrates out the covariates. For a continuous treatment, researchers often report these dose-response functions.[2] In particular, under Assumption U, the distributional dose-response function is given by

$$\mathrm{P}\big(Y^*(t) \leq y\big) = \mathbb{E}\big[\mathrm{F}_{Y^*|T^*X}(y|t, X)\big] \tag{S3}$$

Interestingly, this is exactly the same transformation of the conditional distribution as for the counterfactual distribution in Equation (3) in the main text. Moreover, under Assumption U, an unconditional distributional treatment effect of moving from treatment level $t$ to treatment level $t'$ is given by

$$DTE(t, t') = \mathrm{P}\big(Y^*(t) \leq y\big) - \mathrm{P}\big(Y^*(t') \leq y\big)$$

One can analogously define unconditional quantile treatment effects by the change in particular quantiles when moving from one treatment level to another. Under Assumption U, each of these treatment effect parameters has a causal interpretation.

## SC Life-Cycle Measurement Error

In this appendix, we show how to adapt the life-cycle measurement error framework of Jenkins (1987), Haider and Solon (2006), and Nybom and Stuhler (2016) (among others) to our setting by building on work in Hu and Sasaki (2015) and An, Wang, and Xiao (2020).

We define $Y_{a_Y,i}$ to be child's income if they were observed at age $a_Y$; likewise, we define $T_{a_T,i}$ to be parents' income if they were observed at age $a_T$. Life-cycle measurement error models imply that we can

[2]Dose-response functions are closely related to unconditional quantile treatment effects that are commonly reported in the case of a binary treatment. For example, in the case of intergenerational income mobility, having an income at a high conditional quantile (e.g., conditional on parents' education) indicates that a child has a relatively high income conditional on their parents' income and their parents' education. If child's income tends to be increasing in parents' education, then it could be the case that a child of highly educated parents might be in a low conditional quantile but a middle or upper unconditional quantile. Unconditional quantiles correspond to being in the lower or upper part of the overall income distribution (Frolich and Melly (2013) and Powell (2016) contain good discussions of the difference between conditional and unconditional quantiles).

write

$$Y_{a_Y,i} = \lambda^Y_{a_Y} Y^*_i + U^Y_{a_Y,i}$$
$$T_{a_T,i} = \lambda^T_{a_T} T^*_i + U^T_{a_T,i}$$

where, for conciseness, the notation for the measurement error terms is slightly modified relative to what it was in the main text. Importantly, this allows for systematically different observed incomes depending on the age at which an individual is observed. For example, it is often argued that $\lambda^Y_{a_Y}$ and $\lambda^T_{a_T}$ are less than one for those observed during their early adult years.

Let $A_Y$ denote the age at which child's income is actually observed, and let $A_T$ denote the age at which parents' income is actually observed. Thus, the observed $Y_i = Y_{A_Y,i}$ and $T_i = T_{A_T,i}$. Moreover, we can write

$$Y_{A_Y,i} = \lambda^Y_{A_Y} Y^*_i + U^Y_{A_Y,i}$$
$$T_{A_T,i} = \lambda^T_{A_T} T^*_i + U^T_{A_T,i}$$

We use $\mathcal{A}$ to denote the set of all possible ages (for simplicity, we'll assume that this is the same for parents and children), and we denote the vector of all measurement errors (i.e., across all possible ages) by $U^Y_{\vec{a}_Y}$, for children, and by $U^T_{\vec{a}_T}$ for parents. We make the following assumptions:

**Assumption S1.** $(Y^*, T^*, U^Y_{\vec{a}_Y}, U^T_{\vec{a}_T}, X) \perp\!\!\!\perp (A_Y, A_T)$

**Assumption S2.** *For all* $(a_Y, a_T) \in \mathcal{A}^2$, $(U^Y_{a_Y}, U^T_{a_T}) \perp\!\!\!\perp (Y^*, T^*, X)$

**Assumption S3.** *For all* $(a_Y, a_T) \in \mathcal{A}^2$, $U^Y_{a_Y} \perp\!\!\!\perp U^T_{a_T}$ *and* $(U^Y_{a_Y}/\lambda^Y_{a_Y}) \sim \mathrm{F}_{\tilde{U}_Y}$

**Assumption S4.** *There exist a known* $a^*_Y \in \mathcal{A}$ *and a known* $a^*_T \in \mathcal{A}$ *such that* $\lambda^Y_{a^*_Y} = \lambda^T_{a^*_T} = 1$.

**Assumption S5.** *For all* $(a_Y, a_T) \in \mathcal{A}^2$, $\mathbb{E}[U^Y_{a_Y}] = \mathbb{E}[U^T_{a_T}] = 0$

Assumption S1 says that child's permanent income, parents' permanent income, covariates, and the vector of measurement errors across ages are independent of the age at which income is observed. It coincides with the idea that, in terms of permanent incomes, measurement error, and covariates, differences between cohorts at different ages are due to their being observed at different points in the life-cycle. Assumption S2 says that child's permanent income, parents' permanent income, and covariates are independent of the measurement errors at any age. This is the analog of Assumption 1(ii) in the main text. Assumption S2 allows for the distributions of $U^Y_{a_Y}$ and $U^T_{a_T}$ to vary across $a_Y$ and $a_T$, so that, for example, the variance of the measurement error for parents' income could change at different ages. It also allows for serial correlation in the measurement error so that shocks to income can have persistent effects. Assumption S3 says that the measurement error for child's income is independent of the measurement error for parents' income. This is analogous to Assumption 1(iii) in the main text. The second part says that, after dividing the measurement error for child's income at age $a_Y$ by $\lambda^Y_{a_Y}$, that term follows the same distribution over time. In other words, after accounting for the life-cycle measurement error from $\lambda^Y_{a_Y}$, the distribution of measurement error is the same across ages for children. Alternatively, this assumption could be imposed on the measurement error for parents' income. Assumption S4 says that there exist known ages where the mean of $Y_{a_Y}$ is equal to the mean of $Y^*$ and the mean of $T_{a_T}$ is equal to the mean of $T^*$ — and, therefore, in these periods the classical measurement error conditions hold. This condition is a required normalization in the life-cycle measurement error literature (see, for example, Assumption 1 in An, Wang, and Xiao (2020)), and it is typically thought to be satisfied for individuals in their early to mid-thirties (Haider and Solon (2006) and Nybom and Stuhler (2016)); and, in some cases, it appears to be satisfied for older ages as well. Assumption S5 says that, for all ages, the mean of the measurement errors is equal to 0. Combined with Assumption S1, it implies that $\mathbb{E}[U_{a_Y}^Y|A_Y] = \mathbb{E}[U_{a_T}^T|A_T] = 0$ for all $(a_Y, a_T) \in \mathcal{A}^2$.

**Proposition S1.** *Under Assumptions S1 to S5, for all $a_Y \in \mathcal{A}$ and $a_T \in \mathcal{A}$,*

$$\lambda_{a_Y}^Y = \frac{\mathbb{E}[Y|A_Y = a_Y]}{\mathbb{E}[Y|A_Y = a_Y^*]} \quad \textit{and} \quad \lambda_{a_T}^T = \frac{\mathbb{E}[T|A_T = a_T]}{\mathbb{E}[T|A_T = a_T^*]}$$

*Proof.* We prove the result for $\lambda_{a_Y}^Y$ and note that the result for $\lambda_{a_T}^T$ holds using analogous arguments. For any $a_Y \in \mathcal{A}$,

$$\begin{aligned} \mathbb{E}[Y|A_Y = a_Y] &= \lambda_{a_Y}^Y \mathbb{E}[Y^*|A_Y = a_Y] \\ &= \lambda_{a_Y}^Y \mathbb{E}[Y^*] \end{aligned} \tag{S4}$$

where the first equality holds by Assumptions S1 and S5, and the second equality holds by Assumption S1. Equation (S4) combined with Assumption S4, implies that

$$\mathbb{E}[Y|A_Y = a_Y^*] = \mathbb{E}[Y^*] \tag{S5}$$

Dividing Equation (S4) by Equation (S5) implies the result. □

Given the result in Proposition S1 that $\lambda_{a_Y}^Y$ and $\lambda_{a_T}^T$ are identified, we next define

$$\begin{aligned} \tilde{Y}_i &:= \frac{Y_{A_Y,i}}{\lambda_{A_Y}^Y} = Y_i^* + \tilde{U}_{A_Y,i}^Y \\ \tilde{T}_i &:= \frac{T_{A_T,i}}{\lambda_{A_T}^T} = T_i^* + \tilde{U}_{A_T,i}^T \end{aligned}$$

where $\tilde{U}_{A_Y,i}^Y := \frac{U_{A_Y,i}^Y}{\lambda_{A_Y}^Y}$ and $\tilde{U}_{A_T,i}^T := \frac{U_{A_T,i}^T}{\lambda_{A_T}^T}$. For children, $\tilde{Y}_i$ comes from dividing observed child's income by $\lambda_{a_Y}^Y$ corresponding to the age at which child's income is actually observed; likewise, for parents, $\tilde{T}_i$ comes from dividing observed parents' incomes by $\lambda_{a_T}^T$ corresponding to the age at which parents' income is actually observed.

In the next result, we show that, under the conditions in this section, $\tilde{U}_{A_Y}^Y$ and $\tilde{U}_{A_T}^T$ (the transformed versions of the measurement errors) meet the same requirements imposed on the measurement errors as in the main text in Assumption 1. This implies that, in the context of life-cycle measurement error, $\tilde{Y}_i$ and $\tilde{T}_i$ can be used in place of $Y_i$ and $T_i$ to estimate the nonlinear measures of intergenerational mobility considered in the main text.

**Proposition S2.** *Under Assumptions S1 to S5,*

$$(\tilde{U}_{A_Y}^Y, \tilde{U}_{A_T}^T) \perp\!\!\!\perp (Y^*, T^*, X) \quad \textit{and} \quad \tilde{U}_{A_Y}^Y \perp\!\!\!\perp \tilde{U}_{A_T}^T$$

*Proof.* For any two functions $g$ and $h$ such that the expectation exists, notice that

$$\begin{aligned}
&\mathbb{E}\big[g(\tilde{U}^Y_{A_Y}, \tilde{U}^T_{A_T})h(Y^*, T^*, X)\big] \\
&\quad = \sum_{a_Y \in \mathcal{A}} \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[g\left(\frac{U^Y_{a_Y}}{\lambda^Y_{a_y}}, \frac{U^T_{a_T}}{\lambda^T_{a_T}}\right) h(Y^*, T^*, X) \Big| A_Y = a_Y, A_T = a_T\right] \mathrm{P}(A_Y = a_y, A_T = a_t) \\
&\quad = \sum_{a_Y \in \mathcal{A}} \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[g\left(\frac{U^Y_{a_Y}}{\lambda^Y_{a_y}}, \frac{U^T_{a_T}}{\lambda^T_{a_T}}\right) h(Y^*, T^*, X)\right] \mathrm{P}(A_Y = a_y, A_T = a_t) \\
&\quad = \mathbb{E}\big[h(Y^*, T^*, X)\big] \sum_{a_Y \in \mathcal{A}} \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[g\left(\frac{U^Y_{a_Y}}{\lambda^Y_{a_y}}, \frac{U^T_{a_T}}{\lambda^T_{a_T}}\right)\right] \mathrm{P}(A_Y = a_y, A_T = a_t) \\
&\quad = \mathbb{E}\big[h(Y^*, T^*, X)\big] \sum_{a_Y \in \mathcal{A}} \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[g\left(\frac{U^Y_{a_Y}}{\lambda^Y_{a_y}}, \frac{U^T_{a_T}}{\lambda^T_{a_T}}\right) \Big| A_Y = a_Y, A_T = a_T\right] \mathrm{P}(A_Y = a_y, A_T = a_t) \\
&\quad = \mathbb{E}\big[h(Y^*, T^*, X)\big] \mathbb{E}\left[g\left(\tilde{U}^Y_{A_Y}, \tilde{U}^T_{A_T}\right)\right]
\end{aligned}$$

where the first equality holds by the law of iterated expectations, the second equality holds by Assumption S1, the third equality holds by Assumption S2 and because $\mathbb{E}\big[h(Y^*, T^*, X)\big]$ does not depend on $a_Y$ or $a_T$, the fourth equality holds by Assumption S1, and the last equality holds by the law of iterated expectations. This proves the first part of the result.

For the second part of the result, for any two functions $g$ and $h$ such that the expectation exists, notice that

$$\begin{aligned}
\mathbb{E}\left[g\left(\tilde{U}^Y_{A_Y}\right) h\left(\tilde{U}^T_{A_T}\right)\right] &= \sum_{a_Y \in \mathcal{A}} \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[g\left(\frac{U^Y_{a_Y}}{\lambda^Y_{a_y}}\right) h\left(\frac{U^T_{a_T}}{\lambda^T_{a_T}}\right) \Big| A_Y = a_Y, A_T = a_T\right] \mathrm{P}(A_Y = a_y, A_T = a_T) \\
&= \sum_{a_Y \in \mathcal{A}} \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[g\left(\frac{U^Y_{a_Y}}{\lambda^Y_{a_y}}\right) h\left(\frac{U^T_{a_T}}{\lambda^T_{a_T}}\right)\right] \mathrm{P}(A_Y = a_y, A_T = a_T) \\
&= \sum_{a_Y \in \mathcal{A}} \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[g\left(\frac{U^Y_{a_Y}}{\lambda^Y_{a_y}}\right)\right] \mathbb{E}\left[h\left(\frac{U^T_{a_T}}{\lambda^T_{a_T}}\right)\right] \mathrm{P}(A_Y = a_y, A_T = a_T) \\
&= \sum_{a_Y \in \mathcal{A}} \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[g\left(\tilde{U}^Y_{A_Y}\right)\right] \mathbb{E}\left[h\left(\frac{U^T_{a_T}}{\lambda^T_{a_T}}\right)\right] \mathrm{P}(A_Y = a_y, A_T = a_T) \\
&= \mathbb{E}\left[g\left(\tilde{U}^Y_{A_Y}\right)\right] \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[h\left(\frac{U^T_{a_T}}{\lambda^T_{a_T}}\right)\right] \mathrm{P}(A_T = a_T) \\
&= \mathbb{E}\left[g\left(\tilde{U}^Y_{A_Y}\right)\right] \sum_{a_T \in \mathcal{A}} \mathbb{E}\left[h\left(\frac{U^T_{a_T}}{\lambda^T_{a_T}}\right) \Big| A_T = a_T\right] \mathrm{P}(A_T = a_T) \\
&= \mathbb{E}\left[g\left(\tilde{U}^Y_{A_Y}\right)\right] \mathbb{E}\left[h\left(\tilde{U}^T_{A_T}\right)\right]
\end{aligned}$$

where the first equality holds by the law of iterated expectations, the second equality holds by Assumption S1, the third equality holds by the first part of Assumption S3, the fourth equality holds by the second part of Assumption S3, the fifth equality holds because $\mathbb{E}\big[g(\tilde{U}^Y_{A_Y})\big]$ can be moved outside of the summations and from marginalizing the joint distribution of $A_Y$ and $A_T$, the sixth equality holds by Assumption S1, and the last equality holds by the law of iterated expectations. □

# SD Additional Empirical Results

This appendix contains additional results for the application. In particular, we provide the same results as in the main text but (i) for an alternative copula and (ii) for an alternative set of covariates.

### Frank Copula and Laplace Measurement Error

This section reports the same results as in the main text, but using a Frank copula and Laplace measurement error rather than a Gaussian copula and Gaussian mixture measurement error.[3] First, in this case, we estimate the rank-rank correlation of son's and father's permanent incomes to be 0.366 (s.e.=0.080), which is slightly larger in magnitude (indicating somewhat less intergenerational income mobility) than in the main text.

Table S1: Transition Matrix using a Frank Copula and Laplace Measurement Error

| | | Father's Income Quartile | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Son's Income Quartile | 4 | 0.113 | 0.183 | 0.281 | 0.424 |
| | | (0.026) | (0.019) | (0.006) | (0.041) |
| | 3 | 0.184 | 0.248 | 0.287 | 0.282 |
| | | (0.019) | (0.003) | (0.015) | (0.006) |
| | 2 | 0.280 | 0.289 | 0.248 | 0.183 |
| | | (0.005) | (0.015) | (0.003) | (0.019) |
| | 1 | 0.423 | 0.281 | 0.185 | 0.111 |
| | | (0.042) | (0.005) | (0.020) | (0.025) |

*Notes:* The table provides estimates of a transition matrix allowing for measurement error as in the main text, but using a Frank copula and Laplace measurement error rather than a Gaussian copula. The columns are organized by quartiles of father's income; e.g., columns labeled "1" use data from fathers whose income is in the first quartile. Similarly, rows are organized by quartiles of son's income. Standard errors are computed using the bootstrap.

Next, Table S1 reports our estimate of the transition matrix using a Frank copula and Laplace measurement error. These results are broadly similar to the ones in the main text.

Next, Table S2 provides estimates of upward mobility by quartiles of father's permanent income. Again, these results are broadly similar to those in the main text. Relative to the main text estimates, using a Frank copula and Laplace measurement error, sons whose fathers were in the lower quartiles of the permanent income distribution are somewhat less likely to have higher permanent income ranks than their fathers, while sons whose fathers were in the upper quartiles are somewhat more likely. Otherwise, the estimates are very similar.

Finally, Figure S1 provides estimates of the conditional quantiles of son's permanent income and the

[3] We use a Frank copula with Laplace measurement error in this section because this specification most commonly ranked first by AIC and BIC across several model selection runs. We estimated all combinations of four copula families (Gaussian, Frank, Gumbel, and Clayton), Gaussian mixture measurement error with one or two components for each income equation, and Laplace measurement error (20 specifications in total) and ranked them by AIC and BIC. The specification that we used in the main text, a Gaussian copula with a two-component Gaussian mixture measurement error, was also one of the best specifications by the same criteria and was more stable across all runs.

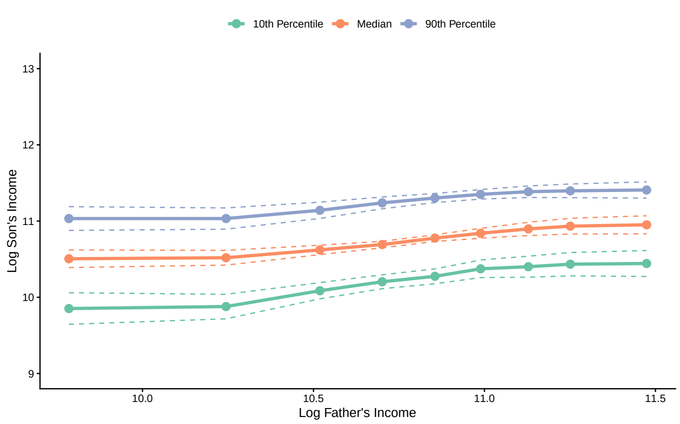
Table S2: Upward Mobility using a Frank Copula and Laplace Measurement Error

| | Father's Income Quartile | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| Measurement Error | | | | |
| | 0.784 | 0.576 | 0.422 | 0.207 |
| | (0.021) | (0.011) | (0.012) | (0.021) |

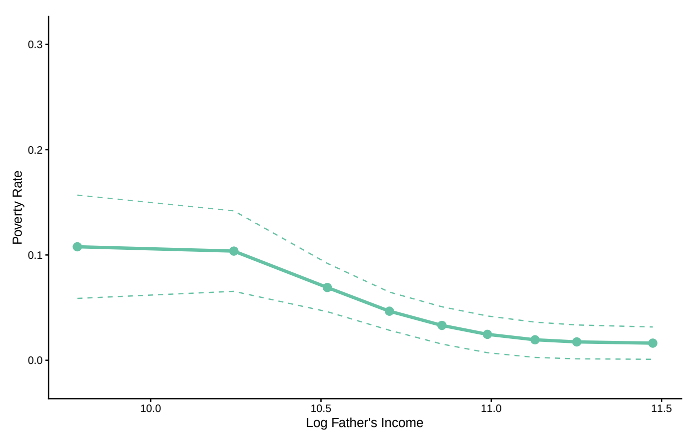
*Notes:* The table provides estimates of upward mobility allowing for measurement error as in the main text but using a Frank copula and Laplace measurement error rather than a Gaussian copula. The columns are organized by quartiles of father's income; e.g., columns labeled "1" use data from fathers whose income is in the first quartile. Standard errors are computed using the bootstrap.

poverty rate, both as a function of father's permanent income. Relative to the results in the main text, the conditional quantile estimates using a Frank copula and Laplace measurement error are broadly similar at low values of father's income. At high values of father's income, the conditional quantiles show somewhat less upward shifting than in the main text. The estimated poverty rate at low values of father's permanent income is also lower under this specification than in the main text, though qualitatively both indicate substantially higher poverty rates for sons with low-income fathers than quantile regression with the observed data.

Figure S1: Quantiles and Poverty Rates for Sons using Frank Copula and Laplace Measurement Error




(a) Quantiles of Son's Income as a Function of Father's Income




(b) Poverty Rate as a Function of Father's Income

*Notes:* The figure provides (i) estimates of quantiles of son's income as a function of father's income and (ii) estimates of the poverty rate of sons as a function of father's income. In both cases, estimates are provided for the 10th, 20th, . . . , and 90th percentiles of observed father's income. Both estimates are conditional on son's age and father's age being equal to their averages in the sample and account for measurement error using the approach suggested in the paper. The difference relative to the estimates in the main text is that a Frank copula is used and measurement error is modeled as Laplace rather than as a Gaussian mixture. Standard errors are computed using the bootstrap.

### Additional Covariates

Finally, we provide results that additionally include son's race and father's years of education as additional covariates in the first step quantile regressions. In this section, we only report results for the copula-type parameters, which are directly comparable to the estimates in the main text; the conditional distribution-type parameters are not comparable because they condition on different sets of covariates, and, therefore, we do not report them in this section. Besides using a different set of covariates, the estimates in this section come from the same estimation procedure in the main text; most notably, as in the main text, we use a Gaussian copula and specify the measurement error as a mixture of two normals.

First, our estimate of the rank-rank correlation in this case is 0.274 (s.e.=0.078). This is somewhat smaller than the corresponding estimate in the main text and is between that estimate and our estimate when we ignore measurement error.

Table S3: Transition Matrix including Additional Covariates

| | | Father's Income Quartile | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Son's Income Quartile | 4 | 0.118 | 0.192 | 0.278 | 0.411 |
| | | (0.030) | (0.013) | (0.008) | (0.036) |
| | 3 | 0.241 | 0.273 | 0.262 | 0.224 |
| | | (0.015) | (0.006) | (0.009) | (0.010) |
| | 2 | 0.296 | 0.266 | 0.241 | 0.197 |
| | | (0.010) | (0.008) | (0.004) | (0.014) |
| | 1 | 0.346 | 0.269 | 0.219 | 0.167 |
| | | (0.039) | (0.006) | (0.015) | (0.028) |

*Notes:* The table provides estimates of a transition matrix allowing for measurement error as in the main text, but where the first step quantile regressions additionally include son's race and father's education as covariates. The columns are organized by quartiles of father's income; e.g., columns labeled "1" use data from fathers whose income is in the first quartile. Similarly, rows are organized by quartiles of son's income. Standard errors are computed using the bootstrap.

Next, Table S3 reports a transition matrix. These results are broadly similar to the ones reported in the main text. These results suggest somewhat more intergenerational mobility than the corresponding estimates in the main text. The main difference is that we estimate that sons whose father's permanent income is in the first quartile are somewhat less likely to stay in the first quartile of the permanent income distribution, though more likely to stay in the first quartile than in the case where we ignore measurement error. The other estimates, particularly for sons whose father's permanent income is in the top quartile, are similar to the results in the main text.

Finally, Table S4 reports upward mobility estimates as a function of father's permanent income quartile. These estimates are broadly similar to the estimates reported in the main text.

Table S4: Upward Mobility including Additional Covariates

| | Father's Income Quartile | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| Measurement Error | | | | |
| | 0.830 | 0.599 | 0.414 | 0.239 |
| | (0.019) | (0.008) | (0.012) | (0.020) |

*Notes:* The table provides estimates of upward mobility allowing for measurement error as in the main text, but where the first step quantile regressions additionally include son's race and father's education as covariates. The columns are organized by quartiles of father's income; e.g., columns labeled "1" use data from fathers whose income is in the first quartile. Standard errors are computed using the bootstrap.

# SE Additional Monte Carlo Simulations

## Laplace Measurement Error

These simulations use the same DGP as in Appendix D except that the measurement error is drawn from a Laplace distribution rather than a Gaussian distribution. The Laplace distribution is ordinary-smooth, whereas the Gaussian is super-smooth. We fix the standard deviation of the measurement error to 0.5 so that the results are directly comparable to the rows with ME s.d. = 0.5 in Table 4. Theoretically, ordinary-smooth measurement error is easier to identify; however, for our simulations, the performance of our estimator is roughly the same (if anything, slightly worse) with Laplace measurement error compared to what it was with Gaussian measurement error.

Table S5: Monte Carlo Simulations with Laplace Measurement Error

| | QR Coef. | | Cop. Param | | Transition Mat. | | |
|---|---|---|---|---|---|---|---|
| ME s.d. | ME | No ME | ME | No ME | ME | No ME | Obs. |
| $n = 250$, **Gaussian Copula** $(\rho = 0.5)$ | | | | | | | |
| 0.5 | 0.334 | 0.320 | 0.111 | 0.092 | 0.039 | 0.027 | 0.050 |
| $n = 1000$, **Gaussian Copula** $(\rho = 0.5)$ | | | | | | | |
| 0.5 | 0.228 | 0.210 | 0.072 | 0.061 | 0.030 | 0.020 | 0.033 |
| $n = 250$, **Clayton Copula** $(\delta = 1.5)$ | | | | | | | |
| 0.5 | 0.332 | 0.317 | 0.413 | 0.629 | 0.028 | 0.048 | 0.057 |
| $n = 1000$, **Clayton Copula** $(\delta = 1.5)$ | | | | | | | |
| 0.5 | 0.229 | 0.210 | 0.254 | 0.545 | 0.017 | 0.042 | 0.043 |

*Notes:* The table reports root mean squared error for (i) estimates of quantile regression parameters (columns labeled "QR Coef"), (ii) estimates of the conditional copula parameter (columns labeled "Cop. Param"), and (iii) estimates of the transition matrix (columns labeled "Transition Mat."). The results in this table differ from those in Table 4 by having measurement error drawn from a Laplace distribution rather than a Gaussian distribution. The column labeled "ME s.d." refers to the standard deviation of the measurement error for $Y$ and $T$ (which are set to be equal to each other). Columns labeled "ME" use the approach suggested in the paper; columns labeled "No ME" ignore measurement error but otherwise follow the same estimation strategy, consisting of first step quantile regressions followed by estimating the conditional copula; the column labeled "Obs." directly uses the observed data to calculate the unconditional transition matrix. The rows differ by the number of observations and the particular conditional copula used. The results are based on 1000 Monte Carlo simulations.

### Correlated Measurement Error

This section examines how the performance of our estimator is affected when Assumption 1(iii), that $U_{Y^*}$ and $U_{T^*}$ are independent, is violated. These are simulations aimed at providing a rough understanding of the sensitivity of our estimator to small deviations from the independence assumption. We use the same DGP as in Appendix D in the appendix to the main text with $n = 1000$, a Gaussian copula with $\rho = 0.5$, and Gaussian measurement error with standard deviation 0.5, but vary $\text{Corr}(U_{Y^*}, U_{T^*})$ from $-0.5$ to 0.7. Table S6 reports the results.

Before discussing the results, it is worth giving a heuristic explanation of what will happen to our estimator when the measurement errors are correlated. First, the quantile regression estimates should be unaffected since they depend only on the marginal distributions of $U_{Y^*}$ and $U_{T^*}$, not on their correlation. To understand the behavior of the copula parameter estimates, consider a simplified case in which $(Y^*, T^*, U_{Y^*}, U_{T^*})$ are jointly normal. By Assumption 1(ii), $\text{Cov}(Y, T) = \text{Cov}(Y^*, T^*) + \text{Cov}(U_{Y^*}, U_{T^*})$. Rearranging in terms of correlations:

$$\text{Corr}(Y^*, T^*) = \underbrace{\frac{\text{sd}(Y)\,\text{sd}(T)}{\text{sd}(Y^*)\,\text{sd}(T^*)}}_{>1} \text{Corr}(Y, T) - \frac{\text{sd}(U_{Y^*})\,\text{sd}(U_{T^*})}{\text{sd}(Y^*)\,\text{sd}(T^*)} \text{Corr}(U_{Y^*}, U_{T^*}).$$

where the underlined term is greater than one because, by Assumption 1(i) and (ii), $Y = Y^* + U_{Y^*}$ with $Y^*$ and $U_{Y^*}$ independent; thus, $\mathrm{Var}(Y) = \mathrm{Var}(Y^*) + \mathrm{Var}(U_{Y^*}) > \mathrm{Var}(Y^*)$, so $\mathrm{sd}(Y) > \mathrm{sd}(Y^*)$ and similarly $\mathrm{sd}(T) > \mathrm{sd}(T^*)$. Under Assumption 1(iii), $\mathrm{Cov}(U_{Y^*}, U_{T^*}) = 0$ and the second term vanishes, and this roughly corresponds to the population version of our estimator by taking $\mathrm{Corr}(Y, T)$ and scaling it up. When Assumption 1(iii) fails, our estimator omits this term, leading to upward bias when $\mathrm{Corr}(U_{Y^*}, U_{T^*}) > 0$ and downward bias when $\mathrm{Corr}(U_{Y^*}, U_{T^*}) < 0$. For the estimator that ignores measurement error (in this simplified setting, corresponding to just estimating the correlation between $Y$ and $T$ using the raw data), from rearranging the previous expression, we have that

$$\mathrm{Corr}(Y,T) = \underbrace{\frac{\mathrm{sd}(Y^*)\,\mathrm{sd}(T^*)}{\mathrm{sd}(Y)\,\mathrm{sd}(T)}}_{<1} \mathrm{Corr}(Y^*, T^*) + \underbrace{\frac{\mathrm{sd}(U_{Y^*})\,\mathrm{sd}(U_{T^*})}{\mathrm{sd}(Y)\,\mathrm{sd}(T)}}_{+} \mathrm{Corr}(U_{Y^*}, U_{T^*}).$$

When $\mathrm{Corr}(U_{Y^*}, U_{T^*}) < 0$, the estimator that uses the raw data will be downward biased. When $\mathrm{Corr}(U_{Y^*}, U_{T^*}) > 0$, two distinct cases arise. If it is "small" and positive, correlated measurement error can reduce the bias of the estimator that ignores measurement error, but if it is large enough, it will make the bias worse than it would have been without correlated measurement error.

The results in Table S6 are broadly consistent with this heuristic. The quantile regression coefficients' RMSE is essentially unchanged across rows, confirming that correlated measurement error does not affect the quantile regression estimates. For the copula parameter, the measurement error-corrected estimator's bias increases monotonically with $\mathrm{Corr}(U_{Y^*}, U_{T^*})$, from $-0.042$ at $\mathrm{Corr} = -0.5$ to 0.221 at $\mathrm{Corr} = 0.7$, consistent with the sign prediction from the first equation above. The estimator that ignores measurement error exhibits the two-part pattern: its bias is most negative when $\mathrm{Corr}(U_{Y^*}, U_{T^*})$ is negative (reaching $-0.165$ at $\mathrm{Corr} = -0.5$), shrinks as the correlation increases, and turns slightly positive for large enough positive correlations (0.059 at $\mathrm{Corr} = 0.7$).

Table S6: Monte Carlo Simulations with Correlated Measurement Error

| | QR Coef. | | Cop. Bias | | Cop. RMSE | | Transition Mat. | | |
|---|---|---|---|---|---|---|---|---|---|
| ME Corr. | ME | No ME | ME | No ME | ME | No ME | ME | No ME | Obs. |
| −0.5 | 0.185 | 0.213 | −0.042 | −0.165 | 0.078 | 0.168 | 0.019 | 0.042 | 0.051 |
| −0.3 | 0.184 | 0.213 | 0.009 | −0.130 | 0.063 | 0.134 | 0.026 | 0.032 | 0.041 |
| −0.1 | 0.183 | 0.212 | 0.053 | −0.091 | 0.078 | 0.095 | 0.023 | 0.033 | 0.042 |
| 0.0 | 0.185 | 0.213 | 0.077 | −0.074 | 0.098 | 0.079 | 0.032 | 0.027 | 0.037 |
| 0.1 | 0.182 | 0.211 | 0.102 | −0.054 | 0.117 | 0.061 | 0.030 | 0.029 | 0.039 |
| 0.3 | 0.182 | 0.211 | 0.142 | −0.018 | 0.152 | 0.033 | 0.047 | 0.018 | 0.029 |
| 0.5 | 0.182 | 0.212 | 0.183 | 0.021 | 0.189 | 0.033 | 0.056 | 0.011 | 0.024 |
| 0.7 | 0.183 | 0.214 | 0.221 | 0.059 | 0.225 | 0.064 | 0.074 | 0.016 | 0.024 |

*Notes:* The table reports results from Monte Carlo simulations using $n = 1000$, a Gaussian copula with $\rho = 0.5$, and Gaussian measurement error with standard deviation 0.5, varying the correlation between the measurement errors $\text{Corr}(U_{Y^*}, U_{T^*})$ across rows. The true copula parameter is $\rho = 0.5$. The columns labeled "QR Coef." report root mean squared error for estimates of the quantile regression parameters. The columns labeled "Cop. Bias" report the mean estimate minus the true value (0.5) for the copula parameter; the columns labeled "Cop. RMSE" report root mean squared error for the same. The columns labeled "Transition Mat." report root mean squared error for estimates of the transition matrix, with "Obs." using the raw observed data directly. Columns labeled "ME" use the approach suggested in the paper; columns labeled "No ME" ignore measurement error. The row with ME Corr. = 0 uses results from Table 4 for comparability. The results are based on 1000 Monte Carlo iterations.

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