Homework 5 Solutions

Hansen 17.2

$\mathbb{E}[e_{it}|X_{it}] = 0$ is not a strong enough condition for $\hat{\beta}$ from a fixed effects regression to be unbiased. This condition says that e_{it} is (mean) independent of X_{it}, but it does not rule out that e_{it} could be related to, say, X_{it+1}. This is not an entirely strange case either, particularly if a good "shock" in the current period leads to the covariate changing in the next time period.

More specifically, recall that

$$\begin{aligned}
\mathbb{E}[\hat{\beta} - \beta|\mathbf{X}] &= \left(\sum_{i=1}^n \dot{\mathbf{X}}_i' \dot{\mathbf{X}}_i \right)^{-1} \sum_{i=1}^n \dot{\mathbf{X}}_i' \mathbb{E}[\mathbf{e}_i|\mathbf{X}] \\
&= \left(\sum_{i=1}^n \dot{\mathbf{X}}_i' \dot{\mathbf{X}}_i \right)^{-1} \sum_{i=1}^n \dot{\mathbf{X}}_i' \mathbb{E}[\mathbf{e}_i|\mathbf{X}_i]
\end{aligned}$$

where $\mathbf{X}$ is the $nT \times k$ "data matrix" and the other notation is from class, and the second equality uses that the observations are independent of each other. Notice that,

$$\mathbb{E}[\mathbf{e}_i|\mathbf{X}_i] = \begin{bmatrix} \mathbb{E}[e_{i1}|X_{i1}, X_{i2}, \dots, X_{iT}] \\ \mathbb{E}[e_{i2}|X_{i1}, X_{i2}, \dots, X_{iT}] \\ \vdots \\ \mathbb{E}[e_{iT}|X_{i1}, X_{i2}, \dots, X_{iT}] \end{bmatrix}$$

The condition in the problem is not strong enough that this term is equal to 0. And, if it is some function of X, then $\hat{\beta}$ would not, in general, be unbiased for β.

Additional Question 1

To start with, notice that

$$\begin{aligned}
\mathbb{E}[\widehat{\text{ATT}}] &= \mathbb{E}\left[\frac{1}{K} \sum_{k=1}^K \widehat{\text{ATT}}(k) \right] \\
&= \frac{1}{K} \sum_{k=1}^K \mathbb{E}\left[\frac{1}{n_k} \sum_{i \in I_k} \left(\frac{D_i}{\pi} - \frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))} \right) (Y_i - \hat{m}_0^{-k}(X_i)) \right] \\
&= \mathbb{E}\left[\left(\frac{D_i}{\pi} - \frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))} \right) (Y_i - \hat{m}_0^{-k}(X_i)) \right]
\end{aligned}$$

where the first equality holds by the definition of $\widehat{\text{ATT}}$, the second equality holds by the definition of $\widehat{\text{ATT}}(k)$, and the third equality holds by the fact that the data are i.i.d.~and that the folds are randomly assigned (which implies that we averaging expectations that are constant across i). Then, we have that

$$
\begin{aligned}
\mathbb{E}[\widehat{\text{ATT}} - \text{ATT}] &= \mathbb{E}\left[\left(\frac{D_i}{\pi} - \frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))}\right)(Y_i - \hat{m}_0^{-k}(X_i))\right] - \mathbb{E}\left[\left(\frac{D_i}{\pi} - \frac{(1-D_i)p(X_i)}{\pi(1-p(X_i))}\right)(Y_i - m_0(X_i))\right] \\
&= -\mathbb{E}\left[\left(\frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))} - \frac{(1-D_i)p(X_i)}{\pi(1-p(X_i))}\right)Y_i\right] \\
&\quad - \mathbb{E}\left[\left(\frac{D_i}{\pi} - \frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))}\right)\hat{m}_0^{-k}(X_i)\right] \\
&\quad + \mathbb{E}\left[\left(\frac{D_i}{\pi} - \frac{(1-D_i)p(X_i)}{\pi(1-p(X_i))}\right)m_0(X_i)\right] \\
&= -\mathbb{E}\left[\left(\frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))} - \frac{(1-D_i)p(X_i)}{\pi(1-p(X_i))}\right)Y_i\right] && \text{(A)} \\
&\quad - \mathbb{E}\left[\left(\frac{D_i}{\pi} - \frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))}\right)\left(\hat{m}_0^{-k}(X_i) - m_0(X_i)\right)\right] && \text{(B)} \\
&\quad + \mathbb{E}\left[\left(\frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))} - \frac{(1-D_i)p(X_i)}{\pi(1-p(X_i))}\right)m_0(X_i)\right] && \text{(C)}
\end{aligned}
$$

where the first equality holds from the previous argument for $\mathbb{E}[\widehat{\text{ATT}}]$ and by using the AIPW estimand for ATT, the second equality holds by rearranging terms and canceling, and the third equality holds by adding and subtracting $\mathbb{E}\left[\left(\frac{D_i}{\pi} - \frac{(1-D_i)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))}\right)m_0(X_i)\right]$ to the second and third lines. Next, notice that

$$
\begin{aligned}
(A) &= -\mathbb{E}\left[\left(\frac{(1-\pi)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))} - \frac{(1-\pi)p(X_i)}{\pi(1-p(X_i))}\right)Y_i \,\middle|\, D_i = 0\right] \\
&= -\mathbb{E}\left[\left(\frac{(1-\pi)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))} - \frac{(1-\pi)p(X_i)}{\pi(1-p(X_i))}\right)m_0(X_i) \,\middle|\, D_i = 0\right]
\end{aligned}
$$

where the second equality holds by the law of iterated expectations. Using, a simlar argument, it follows that

$$
(C) = \mathbb{E}\left[\left(\frac{(1-\pi)\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))} - \frac{(1-\pi)p(X_i)}{\pi(1-p(X_i))}\right)m_0(X_i) \,\middle|\, D_i = 0\right]
$$

Thus, $(A) + (C) = 0$. Finally,

$$
\begin{aligned}
(B) &= -\mathbb{E}\left[\left(\frac{p(X_i)}{\pi} - \frac{(1-p(X_i))\hat{p}^{-k}(X_i)}{\pi(1-\hat{p}^{-k}(X_i))}\right)\left(\hat{m}_0^{-k}(X_i) - m_0(X_i)\right)\right] \\
&= -\frac{1}{\pi}\mathbb{E}\left[\left(\frac{p(X_i)(1-\hat{p}^{-k}(X_i)) - (1-p(X_i))\hat{p}^{-k}(X_i)}{1-\hat{p}^{-k}(X_i)}\right)\left(\hat{m}_0^{-k}(X_i) - m_0(X_i)\right)\right] \\
&= \frac{1}{\pi}\mathbb{E}\left[\frac{(\hat{p}^{-k}(X_i) - p(X_i))(\hat{m}_0^{-k}(X_i) - m_0(X_i))}{1-\hat{p}^{-k}(X_i)}\right]
\end{aligned}
$$

Additional Question 2

Part (a)

```
library(Matrix)
load("../data/job_displacement_clean2.RData")
# drop already treated
data <- subset(data, first.displaced != 2001)
data <- droplevels(data)
Y <- data$learn
data$D <- 1 * ((data$year >= data$first.displaced) & data$first.displaced != 0)
X <- model.matrix(~ as.factor(year) + D, data = data)
n <- length(unique(data$id))
tp <- length(unique(data$year))
iT <- matrix(rep(1, tp))
Dmat <- bdiag(replicate(n, iT, simplify = FALSE))
M <- Matrix::Diagonal(n * tp) - Dmat %*% solve(t(Dmat) %*% Dmat) %*% t(Dmat)
bet <- solve(t(X) %*% M %*% X) %*% t(X) %*% M %*% Y
bet
```

```
8 x 1 Matrix of class "dgeMatrix"
                           [,1]
(Intercept)          5.29358972
as.factor(year)2003  0.09436799
as.factor(year)2005  0.18195412
as.factor(year)2007  0.26904810
as.factor(year)2009  0.28752717
as.factor(year)2011  0.35242722
as.factor(year)2013  0.38427488
D                   -0.23557976
```

Next, let's calculate the standard errors where we use that

$$\begin{aligned}\sqrt{n}(\hat{\beta}-\beta) &= \mathbb{E}[\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i]^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i'\mathbf{M}_i\mathbf{e}_i + o_p(1) \\ &\xrightarrow{d} \mathcal{N}(0,\mathbf{V})\end{aligned}$$

where

$$\begin{aligned}\mathbf{V} &= \mathbb{E}[\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i]^{-1}\boldsymbol{\Omega}\mathbb{E}[\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i]^{-1} \\ &= \mathbb{E}[\dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i]^{-1}\boldsymbol{\Omega}\mathbb{E}[\dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i]^{-1}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\Omega} &= \mathbb{E}[\mathbf{X}_i'\mathbf{M}_i\mathbf{e}_i\mathbf{e}_i'\mathbf{M}_i\mathbf{X}_i] \\ &= \mathbb{E}[\dot{\mathbf{X}}_i'\mathbf{e}_i\mathbf{e}_i'\dot{\mathbf{X}}_i]\end{aligned}$$

It's worth thinking about how to actually estimate these because $\dot{\mathbf{X}}_i$ is a matrix rather than our usual case of it being a vector. First, notice that

$$\dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i = \sum_{t=1}^{T} \dot{X}_{it}\dot{X}_{it}'$$

which is a $k \times k$ matrix. Thus, the natural estimate of $\mathbb{E}[\dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i]$ is

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T} \dot{X}_{it}\dot{X}_{it}' = \dot{\mathbf{X}}'\dot{\mathbf{X}}/n$$

which corresponds to exactly the same way that we estimated this type of term throughout the semester. Next,

$$\dot{\mathbf{X}}_i'\mathbf{e}_i = \sum_{t=1}^{T} X_{it}e_{it}$$

which is a $k \times 1$ vector and so that

$$\dot{\mathbf{X}}_i'\mathbf{e}_i\mathbf{e}_i'\dot{\mathbf{X}}_i = \left(\sum_{t=1}^{T} \dot{X}_{it}e_{it}\right)\left(\sum_{t=1}^{T} \dot{X}_{it}e_{it}\right)'$$

and implies that we would estimate $\mathbf{\Omega}$ by

$$\hat{\mathbf{\Omega}} = \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{t=1}^{T} \dot{X}_{it}\hat{e}_{it}\right)\left(\sum_{t=1}^{T} \dot{X}_{it}\hat{e}_{it}\right)'$$

As far as I know, you can't play the same matrix algebra "trick" that we usually use here (in particular, recall that in the cross sectional case we could estimate $\hat{\mathbf{\Omega}} = \frac{1}{n}\sum_{i=1}^{n} X_iX_i'\hat{e}_i^2$, but that, for programming, it was often convenient to re-express this $\hat{\mathbf{\Omega}} = \tilde{\mathbf{X}}'\tilde{\mathbf{X}}/n$ where a typical element of $\tilde{\mathbf{X}}$ is given by $X_i\hat{e}_i$.) Anyway, the line below that uses the `rowsum` function is essentially just manually calculating $\sum_{t=1}^{T} X_{it}\hat{e}_{it}$ and then using matrix algebra below it.

```
ehat <- as.numeric(Y - X %*% bet)
n <- length(unique(data$id))
dotX <- M %*% X
Q <- t(X) %*% dotX / n
dotXe <- rowsum(as.matrix(dotX * ehat), group = data$id)
Omeg <- t(dotXe) %*% dotXe / n

V <- solve(Q) %*% Omeg %*% solve(Q)
se <- sqrt(diag(V)) / sqrt(n)
round(cbind.data.frame(bet = as.numeric(bet), se = se), 4)
```

```
                      bet     se
(Intercept)         5.2936 0.1003
as.factor(year)2003 0.0944 0.0085
as.factor(year)2005 0.1820 0.0098
```

```
as.factor(year)2007  0.2690 0.0108
as.factor(year)2009  0.2875 0.0115
as.factor(year)2011  0.3524 0.0116
as.factor(year)2013  0.3843 0.0124
D                   -0.2356 0.0257
```

Thus, we estimate that job displacement reduces earnings by about 23%. As a check, let's compare this to what we get from `fixest`.

```
library(fixest)
fe_reg <- feols(learn ~ as.factor(year) + D | id, data = data)
summary(fe_reg)
```

```
OLS estimation, Dep. Var.: learn
Observations: 18,928
Fixed-effects: id: 2,704
Standard-errors: IID
                     Estimate Std. Error   t value  Pr(>|t|)
as.factor(year)2003  0.094368   0.010365   9.10433 < 2.2e-16 ***
as.factor(year)2005  0.181954   0.010411  17.47720 < 2.2e-16 ***
as.factor(year)2007  0.269048   0.010458  25.72633 < 2.2e-16 ***
as.factor(year)2009  0.287527   0.010549  27.25585 < 2.2e-16 ***
as.factor(year)2011  0.352427   0.010629  33.15848 < 2.2e-16 ***
as.factor(year)2013  0.384275   0.010761  35.71008 < 2.2e-16 ***
D                   -0.235580   0.015711 -14.99424 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
RMSE: 0.352029     Adj. R2: 0.75681
                 Within R2: 0.107896
```

These appear to be the same (or the same up to possibly a degree of freedom adjustment).

Part (b)

```
# list of time periods
# for simplicity I'm going to convert this to 1,2,3,4,5,6,7...
tlist <- (sort(unique(data$year)) - 1999) / 2
# list of groups (excluding never-treated)
glist <- (sort(unique(data$first.displaced))[-1] - 1999) / 2

# create new variables in updated time scale
data$G <- ifelse(data$first.displaced == 0, 0, (data$first.displaced - 1999) / 2)
data$tp <- (data$year - 1999) / 2



# write a function to compute att(g,t)
# I compute these as averages using weights, but it
```

```r
# is fine to use subsets of data here too.
# @param w weights, used for bootstrap
# @param base_period, allows base period to optionally
#  be fixed at one
compute.attgt <- function(data, w = rep(1, nrow(data)),
                          use_base_period_1 = FALSE) {
  # data frame to store results
  results <- list()
  counter <- 1

  for (this_t in tlist[-1]) {
    for (this_g in glist) {
      base_period <- min(this_t - 1, this_g - 1)
      if (use_base_period_1) base_period <- 1
      G <- 1 * (data$G == this_g)
      U <- 1 * (data$G == 0)
      pre <- 1 * (data$tp == base_period)
      post <- 1 * (data$tp == this_t)
      pg <- weighted.mean(data$G == this_g, w = w)
      pu <- weighted.mean(data$G == 0, w = w)
      ppre <- mean(pre)
      ppost <- mean(post)

      this_attgt <- weighted.mean(data$learn * G * post / pg / ppost, w = w) -
        weighted.mean(data$learn * G * pre / pg / ppre, w = w) -
        (weighted.mean(data$learn * U * post / pu / ppost, w = w) -
          weighted.mean(data$learn * U * pre / pu / ppre, w = w))
      results[[counter]] <- c(attgt = this_attgt, g = this_g, t = this_t)

      counter <- counter + 1
    }
  }

  # convert to data frame
  results <- as.data.frame(do.call("rbind", results))

  results
}

results <- compute.attgt(data)
# print results
round(results[order(results$g, results$t), ], 4)
```

```
     attgt g t
1  -0.2091 2 2
7  -0.1562 2 3
13 -0.1775 2 4
```

```
19 -0.2375 2 5
25 -0.2347 2 6
31 -0.2781 2 7
2   0.0117 3 2
8  -0.1138 3 3
14 -0.1124 3 4
20 -0.1677 3 5
26 -0.1698 3 6
32 -0.0313 3 7
3   0.0726 4 2
9  -0.0641 4 3
15 -0.1989 4 4
21 -0.3070 4 5
27 -0.2000 4 6
33 -0.2506 4 7
4  -0.0128 5 2
10  0.0036 5 3
16 -0.0603 5 4
22 -0.3184 5 5
28 -0.3115 5 6
34 -0.2210 5 7
5   0.0195 6 2
11 -0.1013 6 3
17 -0.0129 6 4
23  0.0565 6 5
29 -0.2505 6 6
35 -0.1633 6 7
6   0.1183 7 2
12 -0.0379 7 3
18 -0.0295 7 4
24  0.0205 7 5
30 -0.1143 7 6
36 -0.2056 7 7
```

Part (c)

```
# function to compute att0
# ret_weights argument optionally returns the underlying
# weights rather than att0
compute.att0 <- function(attgt_results, w = rep(1, nrow(data)),
                         ret_weights = FALSE) {
  # overall attgt weights
  ever_treated <- which(data$G != 0)
  w <- w[ever_treated]
  pg <- sapply(glist, function(g) weighted.mean(data[ever_treated, ]$G == g, w = w))
  maxT <- max(tlist)
  w0 <- function(g, t) {
```

```
    1 * (t >= g) * pg[glist == g] / (maxT - g + 1)
  }
  # add weights to results
  w0gt <- sapply(1:nrow(attgt_results),
    function(i) w0(attgt_results$g[i], attgt_results$t[i]))
  attgt_results$w0 <- w0gt
  # optionally return computed weights
  if (ret_weights) return(attgt_results)
  att0 <- sum(attgt_results$attgt * attgt_results$w0)
  att0
}

att0 <- compute.att0(results)

# bootstrap standard errors
B <- 100
id_list <- unique(data$id)
boot_att0 <- list()
for (b in 1:B) {
  # draw weights from multinomial distribution (this is exactly the same
  # as empirical bootstrap)
  boot_weights <- as.numeric(rmultinom(n = 1, size = n, prob = rep(1 / n, n)))
  this_boot_weights_id <- cbind.data.frame(id = id_list, boot_weights = boot_weights)
  boot_data <- merge(data, this_boot_weights_id, by = "id")
  boot_attgt <- compute.attgt(data, w = boot_data$boot_weights)
  boot_att0[[b]] <- compute.att0(boot_attgt, w = boot_data$boot_weights)
}

boot_att0 <- do.call("rbind", boot_att0)
se <- sd(boot_att0)

round(cbind.data.frame(att0 = att0, se = se), 4)
```

```
     att0     se
1 -0.2111 0.0272
```

Thus, we are estimating a large, negative and statistically significant effect of job displacement.

Part (d)

```
# function to compute event studies
compute.es <- function(attgt_results, w = rep(1, nrow(data))) {
  # event study weights
  eseq <- sort(unique(attgt_results$t - attgt_results$g))
  es_res <- list()
  counter <- 1
  for (e in eseq) {
```

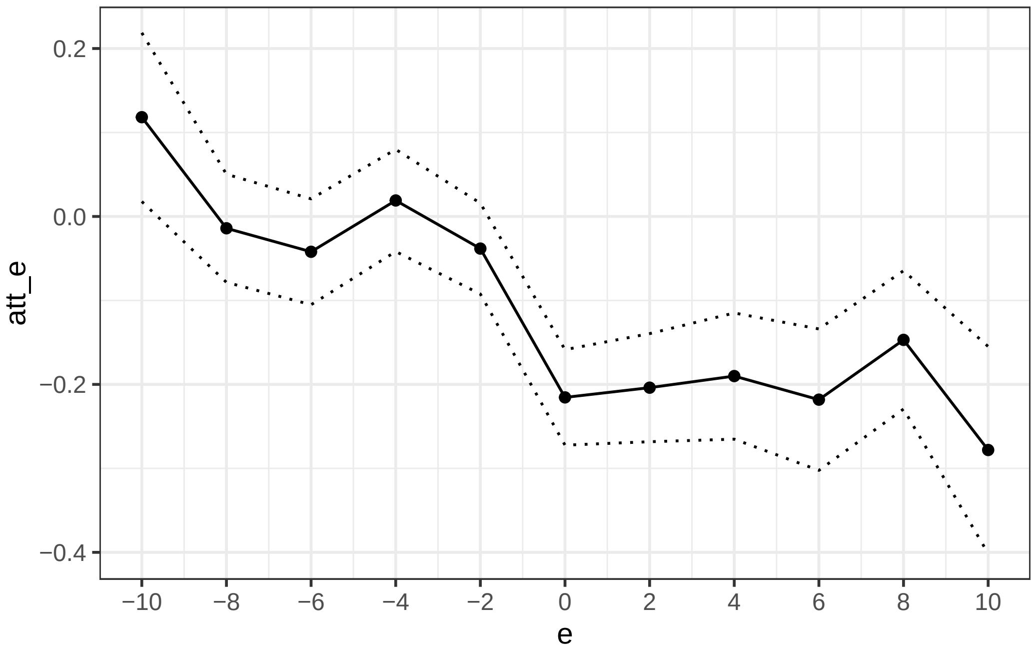
```r
    this_keepers <- which((attgt_results$t - attgt_results$g) == e)
    this_attgt <- attgt_results$attgt[this_keepers]
    pg <- sapply(attgt_results$g[this_keepers],
      function(g) weighted.mean(data$G == g, w = w))
    pg <- pg / sum(pg)
    att_e <- sum(this_attgt * pg)
    es_res[[counter]] <- c(att_e = att_e, e = e)
    counter <- counter + 1
  }
  # convert to data frame
  es_results <- as.data.frame(do.call("rbind", es_res))
  es_results
}

es_results <- compute.es(results)

# bootstrap event study
B <- 100
id_list <- unique(data$id)
boot_es <- list()
for (b in 1:B) {
  boot_weights <- as.numeric(rmultinom(n = 1, size = n, prob = rep(1 / n, n)))
  this_boot_weights_id <- cbind.data.frame(id = id_list, boot_weights = boot_weights)
  boot_data <- merge(data, this_boot_weights_id, by = "id")
  boot_attgt <- compute.attgt(data, w = boot_data$boot_weights)
  boot_es[[b]] <- compute.es(boot_attgt, w = boot_data$boot_weights)$att_e
}

boot_es <- do.call("rbind", boot_es)
se <- apply(boot_es, 2, sd)
es_results$se <- se
es_results$ciL <- es_results$att_e - 1.96 * es_results$se
es_results$ciU <- es_results$att_e + 1.96 * es_results$se

library(ggplot2)
ggplot(data = es_results, mapping = aes(x = e, y = att_e)) +
  geom_line() +
  geom_point(size = 1.5) +
  geom_line(aes(y = ciU), linetype = "dotted") +
  geom_line(aes(y = ciL), linetype = "dotted") +
  scale_x_continuous(breaks = seq(-5, 5), labels = seq(-10, 10, 2)) +
  theme_bw()
```



The figure suggests that job displacement causes earnings to drop by, on average, about 20% and that this effect is quite persistent; it appears to be roughly the same 10 years following job displacement. If you look at the estimates in pre-treatment periods, with the exception of 10 years before job displacement, the estimates are fairly close to 0 (and not statistically different from 0) suggesting that the parallel trends assumption is likely to be fairly reasonable in this application.