# Homework 6 Solutions

## Hansen 17.2

$\mathbb{E}[e_{it}|X_{it}] = 0$ is not a strong enough condition for $\hat{\beta}$ from a fixed effects regression to be unbiased. This condition says that $e_{it}$ is (mean) independent of $X_{it}$, but it does not rule out that $e_{it}$ could be related to, say, $X_{it+1}$. This is not an entirely strange case either, particularly if a good "shock" in the current period leads to the covariate changing in the next time period.

More specifically, recall that

$$\begin{aligned}\mathbb{E}[\hat{\beta} - \beta|\mathbf{X}] &= \left(\sum_{i=1}^{n} \dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i\right)^{-1} \sum_{i=1}^{n} \dot{\mathbf{X}}_i'\mathbb{E}[\mathbf{e}_i|\mathbf{X}] \\ &= \left(\sum_{i=1}^{n} \dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i\right)^{-1} \sum_{i=1}^{n} \dot{\mathbf{X}}_i'\mathbb{E}[\mathbf{e}_i|\mathbf{X}_i]\end{aligned}$$

where $\mathbf{X}$ is the $nT \times k$ "data matrix" and the other notation is from class, and the second equality uses that the observations are independent of each other. Notice that,

$$\mathbb{E}[\mathbf{e}_i|\mathbf{X}_i] = \begin{bmatrix} \mathbb{E}[e_{i1}|X_{i1}, X_{i2}, \dots, X_{iT}] \\ \mathbb{E}[e_{i2}|X_{i1}, X_{i2}, \dots, X_{iT}] \\ \vdots \\ \mathbb{E}[e_{iT}|X_{i1}, X_{i2}, \dots, X_{iT}] \end{bmatrix}$$

The condition in the problem is not strong enough that this term is equal to 0. And, if it is some function of $X$, then $\hat{\beta}$ would not, in general, be unbiased for $\beta$.

## Additional Question 1

### Part (a)

```r
library(Matrix)
load("job_displacement_clean2.RData")
# drop already treated
data <- subset(data, first.displaced != 2001)
data <- droplevels(data)
Y <- data$learn
data$D <- 1*( (data$year >= data$first.displaced) & data$first.displaced != 0)
X <- model.matrix(~ as.factor(year) + D, data=data)
n <- length(unique(data$id))
tp <- length(unique(data$year))
iT <- matrix(rep(1,tp))
Dmat <- bdiag(replicate(n,iT,simplify=FALSE))
M <- Matrix::Diagonal(n*tp) - Dmat%*%solve(t(Dmat)%*%Dmat)%*%t(Dmat)
bet <- solve(t(X)%*%M%*%X) %*% t(X)%*%M%*%Y
bet
```

```
8 x 1 Matrix of class "dgeMatrix"
                            [,1]
```

```
(Intercept)           5.29358972
as.factor(year)2003  0.09436799
as.factor(year)2005  0.18195412
as.factor(year)2007  0.26904810
as.factor(year)2009  0.28752717
as.factor(year)2011  0.35242722
as.factor(year)2013  0.38427488
D                   -0.23557976
```

Next, let's calculate the standard errors where we use that

$$\begin{aligned}\sqrt{n}(\hat{\beta} - \beta) &= \mathbb{E}[\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i]^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i'\mathbf{M}_i\mathbf{e}_i + o_p(1) \\ &\xrightarrow{d} \mathcal{N}(0, \mathbf{V})\end{aligned}$$

where

$$\begin{aligned}\mathbf{V} &= \mathbb{E}[\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i]^{-1}\mathbf{\Omega}\mathbb{E}[\mathbf{X}_i'\mathbf{M}_i\mathbf{X}_i]^{-1} \\ &= \mathbb{E}[\dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i]^{-1}\mathbf{\Omega}\mathbb{E}[\dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i]^{-1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{\Omega} &= \mathbb{E}[\mathbf{X}_i'\mathbf{M}_i\mathbf{e}_i\mathbf{e}_i'\mathbf{M}_i\mathbf{X}_i] \\ &= \mathbb{E}[\dot{\mathbf{X}}_i'\mathbf{e}_i\mathbf{e}_i'\dot{\mathbf{X}}_i]\end{aligned}$$

It's worth thinking about how to actually estimate these because $\dot{\mathbf{X}}_i$ is a matrix rather than our usual case of it being a vector. First, notice that

$$\dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i = \sum_{t=1}^{T}\dot{X}_{it}\dot{X}_{it}'$$

which is a $k \times k$ matrix. Thus, the natural estimate of $\mathbb{E}[\dot{\mathbf{X}}_i'\dot{\mathbf{X}}_i]$ is

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\dot{X}_{it}\dot{X}_{it}' = \dot{\mathbf{X}}'\dot{\mathbf{X}}/n$$

which corresponds to exactly the same way that we estimated this type of term throughout the semester. Next,

$$\dot{\mathbf{X}}_i'\mathbf{e}_i = \sum_{t=1}^{T}X_{it}e_{it}$$

which is a $k \times 1$ vector and so that

$$\dot{\mathbf{X}}_i'\mathbf{e}_i\mathbf{e}_i'\dot{\mathbf{X}}_i = \left(\sum_{t=1}^{T}\dot{X}_{it}e_{it}\right)\left(\sum_{t=1}^{T}\dot{X}_{it}e_{it}\right)'$$

and implies that we would estimate $\mathbf{\Omega}$ by

$$\hat{\mathbf{\Omega}} = \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\dot{X}_{it}\hat{e}_{it}\right)\left(\sum_{t=1}^{T}\dot{X}_{it}\hat{e}_{it}\right)'$$

As far as I know, you can't play the same matrix algebra "trick" that we usually use here (in particular, recall that in the cross sectional case we could estimate $\hat{\mathbf{\Omega}} = \frac{1}{n}\sum_{i=1}^{n} X_i X_i' \hat{e}_i^2$, but that, for programming, it was often convenient to re-express this $\hat{\mathbf{\Omega}} = \tilde{\mathbf{X}}'\tilde{\mathbf{X}}/n$ where a typical element of $\tilde{\mathbf{X}}$ is given by $X_i \hat{e}_i$.) Anyway, the line below that uses the `rowsum` function is essentially just manually calculating $\sum_{t=1}^{T} X_{it}\hat{e}_{it}$ and then using matrix algebra below it.

```
ehat <- as.numeric(Y - X%*%bet)
n <- length(unique(data$id))
dotX <- M%*%X
Q <- t(X) %*% dotX / n
dotXe <- rowsum(as.matrix(dotX*ehat), group=data$id)
Omeg <- t(dotXe)%*%dotXe/n

V <- solve(Q)%*%Omeg%*%solve(Q)
se <- sqrt(diag(V))/sqrt(n)
round(cbind.data.frame(bet=as.numeric(bet), se=se), 4)
```

```
                         bet     se
(Intercept)           5.2936 0.1003
as.factor(year)2003   0.0944 0.0085
as.factor(year)2005   0.1820 0.0098
as.factor(year)2007   0.2690 0.0108
as.factor(year)2009   0.2875 0.0115
as.factor(year)2011   0.3524 0.0116
as.factor(year)2013   0.3843 0.0124
D                    -0.2356 0.0257
```

Thus, we estimate that job displacement reduces earnings by about 23%. As a check, let's compare this to what we get from `fixest`.

```
library(fixest)
fe_reg <- feols(learn ~ as.factor(year) + D | id, data=data)
summary(fe_reg)
```

```
OLS estimation, Dep. Var.: learn
Observations: 18,928
Fixed-effects: id: 2,704
Standard-errors: Clustered (id)
                     Estimate Std. Error  t value  Pr(>|t|)
as.factor(year)2003  0.094368   0.008533 11.05873 < 2.2e-16 ***
as.factor(year)2005  0.181954   0.009756 18.65073 < 2.2e-16 ***
as.factor(year)2007  0.269048   0.010758 25.00982 < 2.2e-16 ***
```

```
as.factor(year)2009  0.287527   0.011507 24.98624 < 2.2e-16 ***
as.factor(year)2011  0.352427   0.011598 30.38799 < 2.2e-16 ***
as.factor(year)2013  0.384275   0.012396 30.99875 < 2.2e-16 ***
D                   -0.235580   0.025677 -9.17486 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
RMSE: 0.352029     Adj. R2: 0.75681
                 Within R2: 0.107896
```

These appear to be the same (or the same up to possibly a degree of freedom adjustment).

### Part (b)

```r
# list of time periods
# for simplicity I'm going to convert this to 1,2,3,4,5,6,7...
tlist <- (sort(unique(data$year)) - 1999)/2
# list of groups (excluding never-treated)
glist <- (sort(unique(data$first.displaced))[-1] - 1999)/2

# create new variables in updated time scale
data$G <- ifelse(data$first.displaced==0, 0, (data$first.displaced - 1999)/2)
data$tp <- (data$year-1999)/2


# write a function to compute att(g,t)
# I compute these as averages using weights, but it
# is fine to use subsets of data here too.
# @param w weights, used for bootstrap
# @param base_period, allows base period to optionally
#  be fixed at one
compute.attgt <- function(data, w=rep(1,nrow(data)),
                          use_base_period_1=FALSE) {
  # data frame to store results
  results <- list()
  counter <- 1

  for (this_t in tlist[-1]) {
    for (this_g in glist) {
      base_period <- min(this_t-1, this_g-1)
      if (use_base_period_1) base_period <- 1
      G <- 1*(data$G==this_g)
      U <- 1*(data$G==0)
      pre <- 1*(data$tp == base_period)
      post <- 1*(data$tp == this_t)
      pg <- weighted.mean(data$G == this_g, w=w)
      pu <- weighted.mean(data$G == 0, w=w)
      ppre <- mean(pre)
```

```r
      ppost <- mean(post)

      this_attgt <- weighted.mean(data$learn*G*post/pg/ppost, w=w) -
                          weighted.mean(data$learn*G*pre/pg/ppre, w=w) -
        (weighted.mean(data$learn*U*post/pu/ppost, w=w) -
           weighted.mean(data$learn*U*pre/pu/ppre, w=w))
      results[[counter]] <- c(attgt=this_attgt, g=this_g, t=this_t)

      counter <- counter+1
    }
  }

  # convert to data frame
  results <- as.data.frame(do.call("rbind", results))

  results
}

results <- compute.attgt(data)
# print results
round(results[order(results$g, results$t),],4)
```

```
     attgt g t
1  -0.2091 2 2
7  -0.1562 2 3
13 -0.1775 2 4
19 -0.2375 2 5
25 -0.2347 2 6
31 -0.2781 2 7
2   0.0117 3 2
8  -0.1138 3 3
14 -0.1124 3 4
20 -0.1677 3 5
26 -0.1698 3 6
32 -0.0313 3 7
3   0.0726 4 2
9  -0.0641 4 3
15 -0.1989 4 4
21 -0.3070 4 5
27 -0.2000 4 6
33 -0.2506 4 7
4  -0.0128 5 2
10  0.0036 5 3
16 -0.0603 5 4
22 -0.3184 5 5
28 -0.3115 5 6
34 -0.2210 5 7
```

```
5   0.0195 6 2
11 -0.1013 6 3
17 -0.0129 6 4
23  0.0565 6 5
29 -0.2505 6 6
35 -0.1633 6 7
6   0.1183 7 2
12 -0.0379 7 3
18 -0.0295 7 4
24  0.0205 7 5
30 -0.1143 7 6
36 -0.2056 7 7
```

#### Part (c)

```
# function to compute att0
# ret_weights argument optionally returns the underlying
# weights rather than att0
compute.att0 <- function(attgt_results, w=rep(1,nrow(data)),
                         ret_weights=FALSE) {
  # overall attgt weights
  ever_treated <- which(data$G != 0)
  w <- w[ever_treated]
  pg <- sapply(glist, function(g) weighted.mean(data[ever_treated,]$G==g, w=w))
  maxT <- max(tlist)
  w0 <- function(g,t) {
    1*(t >= g)*pg[glist==g] / (maxT - g + 1)
  }
  # add weights to results
  w0gt <- sapply(1:nrow(attgt_results),
                 function(i) w0(attgt_results$g[i], attgt_results$t[i]))
  attgt_results$w0 <- w0gt
  # optionally return computed weights
  if(ret_weights) return(attgt_results)
  att0 <- sum(attgt_results$attgt*attgt_results$w0)
  att0
}

att0 <- compute.att0(results)

# bootstrap standard errors
B <- 100
id_list <- unique(data$id)
boot_att0 <- list()
for (b in 1:B) {
  # draw weights from multinomial distribution (this is exactly the same
  # as empirical bootstrap)
```

```r
  boot_weights <- as.numeric(rmultinom(n=1, size=n, prob=rep(1/n,n)))
  this_boot_weights_id <- cbind.data.frame(id=id_list, boot_weights=boot_weights)
  boot_data <- merge(data, this_boot_weights_id, by="id")
  boot_attgt <- compute.attgt(data, w=boot_data$boot_weights)
  boot_att0[[b]] <- compute.att0(boot_attgt, w=boot_data$boot_weights)
}

boot_att0 <- do.call("rbind", boot_att0)
se <- sd(boot_att0)

round(cbind.data.frame(att0=att0, se=se), 4)
```

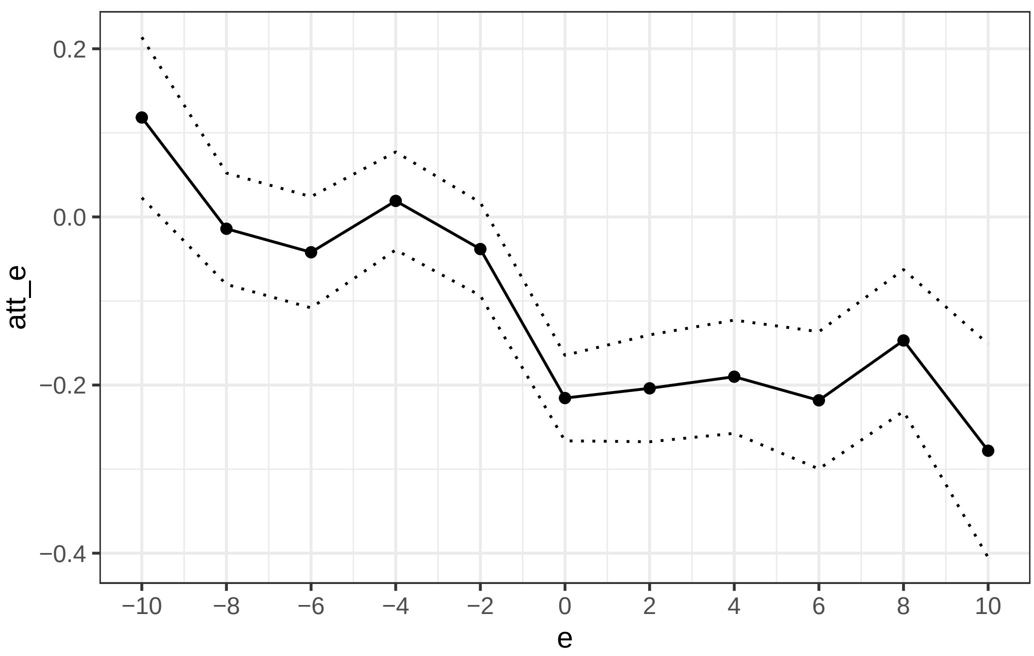
```
     att0     se
1 -0.2111 0.0237
```

Thus, we are estimating a large, negative and statistically significant effect of job displacement.

### Part (d)

```r
# function to compute event studies
compute.es <- function(attgt_results, w=rep(1,nrow(data))) {
  # event study weights
  eseq <- sort(unique(attgt_results$t - attgt_results$g))
  es_res <- list()
  counter <- 1
  for (e in eseq) {
    this_keepers <- which( (attgt_results$t - attgt_results$g) == e)
    this_attgt <- attgt_results$attgt[this_keepers]
    pg <- sapply(attgt_results$g[this_keepers],
                 function(g)  weighted.mean(data$G==g, w=w))
    pg <- pg / sum(pg)
    att_e <- sum(this_attgt*pg)
    es_res[[counter]] <- c(att_e=att_e, e=e)
    counter <- counter+1
  }
  # convert to data frame
  es_results <- as.data.frame(do.call("rbind", es_res))
  es_results
}

es_results <- compute.es(results)

# bootstrap event study
B <- 100
id_list <- unique(data$id)
boot_es <- list()
for (b in 1:B) {
```

```
  boot_weights <- as.numeric(rmultinom(n=1, size=n, prob=rep(1/n,n)))
  this_boot_weights_id <- cbind.data.frame(id=id_list, boot_weights=boot_weights)
  boot_data <- merge(data, this_boot_weights_id, by="id")
  boot_attgt <- compute.attgt(data, w=boot_data$boot_weights)
  boot_es[[b]] <- compute.es(boot_attgt, w=boot_data$boot_weights)$att_e
}

boot_es <- do.call("rbind", boot_es)
se <- apply(boot_es, 2, sd)
es_results$se <- se
es_results$ciL <- es_results$att_e - 1.96*es_results$se
es_results$ciU <- es_results$att_e + 1.96*es_results$se

library(ggplot2)
ggplot(data=es_results, mapping=aes(x=e,y=att_e)) +
  geom_line() +
  geom_point(size=1.5) +
  geom_line(aes(y=ciU), linetype="dotted") +
  geom_line(aes(y=ciL), linetype="dotted") +
  scale_x_continuous(breaks=seq(-5,5), labels=seq(-10,10,2)) +
  theme_bw()
```




The figure suggests that job displacement causes earnings to drop by, on average, about 20% and that this effect is quite persistent; it appears to be roughly the same 10 years following job displacement. If you look at the estimates in pre-treatment periods, with the exception of 10 years before job displacement, the estimates are fairly close to 0 (and not statistically different from 0) suggesting that the parallel trends assumption is likely to be fairly reasonable in this application.

### Additional Question 2

We can rewrite the expression in the problem as

$$\hat{\beta}_{gmm} = \underset{b}{\text{argmin}}\ \mathbf{Y}'\mathbf{Z}\widehat{\mathbf{W}}\mathbf{Z}'\mathbf{Y} - 2b'\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\mathbf{Z}'\mathbf{Y} + b'\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\mathbf{Z}'\mathbf{X}b$$

Taking the first order condition, we have that

$$\begin{aligned} 0 &= -\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\mathbf{Z}'\mathbf{Y} + \mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\mathbf{Z}'\mathbf{X}\hat{\beta}_{gmm} \\ \implies \hat{\beta}_{gmm} &= (\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\mathbf{Z}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\mathbf{Z}'\mathbf{Y} \end{aligned}$$

This completes the first part of the problem. For the asymptotic distribution, notice that we can re-write the previous equation as

$$\begin{aligned} \hat{\beta}_{gmm} &= \left(\frac{1}{n}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\frac{1}{n}\mathbf{Z}'\mathbf{X}\right)^{-1}\frac{1}{n}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\frac{1}{n}\sum_{i=1}^{n} Z_iY_i \\ &= \left(\frac{1}{n}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\frac{1}{n}\mathbf{Z}'\mathbf{X}\right)^{-1}\frac{1}{n}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\frac{1}{n}\sum_{i=1}^{n} Z_i(X_i'\beta + e_i) \\ &= \beta + \left(\frac{1}{n}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\frac{1}{n}\mathbf{Z}'\mathbf{X}\right)^{-1}\frac{1}{n}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\frac{1}{n}\sum_{i=1}^{n} Z_ie_i \end{aligned}$$

This implies that

$$\begin{aligned} \sqrt{n}(\hat{\beta}_{gmm} - \beta) &= \left(\frac{1}{n}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\frac{1}{n}\mathbf{Z}'\mathbf{X}\right)^{-1}\frac{1}{n}\mathbf{X}'\mathbf{Z}\widehat{\mathbf{W}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_ie_i \\ &= \Big(\mathbb{E}[XZ']\mathbf{W}\mathbb{E}[ZX']\Big)^{-1}\mathbb{E}[XZ']\mathbf{W}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_ie_i + o_p(1) \end{aligned}$$

where the second equality holds because $\widehat{\mathbf{W}} \xrightarrow{p} \mathbf{W}$ and because $\frac{1}{n}\mathbf{X}'\mathbf{Z} = \frac{1}{n}\sum_{i=1}^{n} X_iZ_i' \xrightarrow{p} \mathbb{E}[XZ']$ and by the continuous mapping theorem. Next, notice that $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_ie_i \xrightarrow{d} \mathcal{N}(0, \mathbf{\Omega})$ where $\mathbf{\Omega} := \mathbb{E}[ZZ'e^2]$. Thus, by the continuous mapping theorem, we have that,

$$\sqrt{n}(\hat{\beta}_{gmm} - \beta) \xrightarrow{d} \mathcal{N}(0, \mathbf{V})$$

where

$$\mathbf{V} = \Big(\mathbb{E}[XZ']\mathbf{W}\mathbb{E}[ZX']\Big)^{-1}\mathbb{E}[XZ']\mathbf{W}\mathbf{\Omega}\mathbf{W}\mathbb{E}[ZX']\Big(\mathbb{E}[XZ']\mathbf{W}\mathbb{E}[ZX']\Big)^{-1}$$