Homework 1 Solutions

Hansen 2.2

$$\begin{aligned}\mathbb{E}[YX] &= \mathbb{E}[X\mathbb{E}[Y|X]] \\ &= \mathbb{E}[X(a+bX)] \\ &= \mathbb{E}[aX+bX^2] \\ &= a\mathbb{E}[X]+b\mathbb{E}[X^2]\end{aligned}$$

where the first equality holds by the law of iterated expectations, the second equality holds by the expression for $\mathbb{E}[Y|X]$ in the problem, and the remaining two equalities are just algebra/basic properties of expectations.

Hansen 2.5

a) The mean squared error is given by

$$MSE = \mathbb{E}[(e^2 - h(X))^2]$$

b) e^2 is closely related to a measure of the magnitude of how far off our predictions of Y given X are. For example, given X, if we predict a "high" value of e^2, it would suggest that we expect our predictions of Y to not be too accurate for that value of X.

c) Recall that $\sigma^2(X) = \mathbb{E}[e^2|X]$ so that

$$\begin{aligned}MSE &= \mathbb{E}\Big[((e^2 - \mathbb{E}[e^2|X]) - (h(X) - \mathbb{E}[e^2|X]))^2\Big] \\ &= \mathbb{E}\Big[(e^2 - \mathbb{E}[e^2|X])^2\Big] - 2\mathbb{E}\Big[(e^2 - \mathbb{E}[e^2|X])(h(X) - \mathbb{E}[e^2|X])\Big] + \mathbb{E}\Big[(h(X) - \mathbb{E}[e^2|X])^2\Big]\end{aligned}$$

Let's consider each of these three terms.

- The first term does not depend on $h(X)$ so it is invariant to our choice of h.
- The second term is equal to 0 after applying the law of iterated expectations.
- The third term is minimized by setting $h(X) = \mathbb{E}[e^2|X] = \sigma^2(X)$ which implies that MSE is minimized by $\sigma^2(X)$.

Hansen 2.6

To start with, notice that

$$\mathbb{E}[Y] = \mathbb{E}[m(X) + e] = \mathbb{E}[m(X)]$$

where the last equality holds because $\mathbb{E}[e] = 0$. Thus, we have that

$$\begin{aligned}
\text{var}(Y) &= \mathbb{E}\Big[(Y - \mathbb{E}[Y])^2\Big] \\
&= \mathbb{E}\Big[(m(X) + e - \mathbb{E}[m(X)])^2\Big] \\
&= \mathbb{E}\Big[(m(X) - \mathbb{E}[m(X)])^2\Big] + 2\mathbb{E}\Big[(m(X) - \mathbb{E}[m(X)])e\Big] + \mathbb{E}[e^2] \\
&= \mathbb{E}\Big[(m(X) - \mathbb{E}[m(X)])^2\Big] + \mathbb{E}[e^2] \\
&= \text{var}[m(X)] + \sigma^2
\end{aligned}$$

where first equality holds by the definition of $\text{var}(Y)$, the second equality holds by the expression for $\mathbb{E}[Y]$ in the previous display, the third equality holds by expanding the square, the fourth equality holds by applying the law of iterated expectations to the middle term (and because $\mathbb{E}[e|X] = 0$), and the fifth equality holds by the definition of $\text{var}[m(X)]$ and the fact that $\mathbb{E}[e^2] = \sigma^2$.

Hansen 2.10

True.

$$\mathbb{E}[X^2 e] = \mathbb{E}[X^2 \underbrace{\mathbb{E}[e|X]}_{=0}] = 0$$

Hansen 2.11

False. Here is a counterexample. Suppose that $X = 1$ with probability 1/2 and that $X = -1$ with probability 1/2. Importantly, this means that $X^2 = 1$, $X^3 = X$, $X^4 = 1$, and so on; this further implies that $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 1$, $\mathbb{E}[X^3] = 0$ and so on. Also, suppose that $\mathbb{E}[e|X] = X^2$. Then, $\mathbb{E}[Xe] = \mathbb{E}[X\mathbb{E}[e|X]] = \mathbb{E}[X \cdot X^2] = \mathbb{E}[X^3] = 0$. However, $\mathbb{E}[X^2 e] = \mathbb{E}[X^2\mathbb{E}[e|X]] = \mathbb{E}[X^2 \cdot X^2] = \mathbb{E}[X^4] = 1 \neq 0$

Hansen 2.12

False. Here is a counterexample. Suppose that $\mathbb{E}[e^2|X]$ depends on X, then e and X are not independent. As a concrete counterexample, suppose $e|X \sim \mathcal{N}(0, X^2)$ (that is, conditional on X, e follows a normal distribution with mean 0 and variance X^2). In this case $\mathbb{E}[e|X] = 0$, but e and X are not independent.

Hansen 2.13

False. The same counterexample as in 2.11 works here. In that case, $\mathbb{E}[Xe] = 0$, but $\mathbb{E}[e|X] = X^2$ (in that case $X^2 = 1$, but the main point is that it is not equal to 0 for all values of X).

Hansen 2.14

False. In this case, higher order moments can still depend on X. For example, $\mathbb{E}[e^3|X]$ can still depend on X. If it does, then e and X are not independent.

Hansen 2.21

a) Following omitted variable bias types of arguments (also, notice that the notation in the problem implies that X is scalar here), we have that

$$\begin{aligned}\gamma_1 &= \frac{\mathbb{E}[XY]}{\mathbb{E}[X^2]} \\ &= \frac{\mathbb{E}[X(X\beta_1 + X^2\beta_2 + u)]}{\mathbb{E}[X^2]} \\ &= \beta_1 + \frac{\mathbb{E}[X^3]}{\mathbb{E}[X^2]}\beta_2\end{aligned}$$

Thus, $\gamma_1 = \beta_1$ if either $\beta_2 = 0$ or $\mathbb{E}[X^3] = 0$. $\beta_2 = 0$ if X^2 does not have an effect on the outcome (after accounting for the effect of X); this is similar to the omitted variable logic that we talked about in class. A leading case where $\mathbb{E}[X^3] = 0$ is when X is a mean 0 symmetric random variable; for example, if X is standard normal, then its third moment is equal to 0.

b) Using the same arguments as in part (a), we have that

$$\gamma_1 = \theta_1 + \frac{\mathbb{E}[X^4]}{\mathbb{E}[X^2]}\theta_2$$

Similar to the previous part, γ_1 could equal θ_1 if θ_2 were equal to 0. Unlike the previous part though, here, we cannot have that $\mathbb{E}[X^4] = 0$ except in the degenerate case where $X = 0$ with probability 1 (which would be ruled out here as it would also imply that $\mathbb{E}[X^2] = 0$).

Extra Question

```r
load("fertilizer_2000.RData")


# part (a)
nrow(fertilizer_2000)
```

```
[1] 68
```

```r
# part (b)
fertilizer_2000[21,]$country
```

```
[1] "Gambia, The"
```

```r
# part (c)
mean_gdp <- mean(fertilizer_2000$avgdppc)
mean_gdp
```

```
[1] 4291.377
```

```r
# part (d)
above_avg_gdp <- subset(fertilizer_2000, avgdppc > mean_gdp)
mean(above_avg_gdp$prec)
```

```
[1] 1391.391
```