Solutions to Additional Practice Questions

Hansen 3.2

Let's call $\tilde{\beta}$ and $\tilde{\mathbf{e}}$ the OLS estimates and residuals from the regression of $\mathbf{Y}$ on $\mathbf{Z}$. Notice that

$$\begin{aligned}
\tilde{\beta} &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} \\
&= \left((\mathbf{XC})'\mathbf{XC}\right)^{-1}(\mathbf{XC})'\mathbf{Y} \\
&= (\mathbf{C}'\mathbf{X}'\mathbf{XC})^{-1}\mathbf{C}'\mathbf{X}'\mathbf{Y} \\
&= \mathbf{C}^{-1}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}'^{-1}\mathbf{C}'\mathbf{X}'\mathbf{Y} \\
&= \mathbf{C}^{-1}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\
&= \mathbf{C}^{-1}\hat{\beta}
\end{aligned}$$

where the second equality holds by plugging in $\mathbf{Z} = \mathbf{XC}$, the third equality holds by taking the transpose of $\mathbf{XC}$, the fourth equality holds because $\mathbf{C}$ and $\mathbf{X}'\mathbf{X}$ are nonsingular, the fifth equality holds by canceling the $\mathbf{C}'^{-1}\mathbf{C}'$, and the last equality holds by the definition of $\hat{\beta}$.

Now, for the residuals, notice that

$$\begin{aligned}
\tilde{\mathbf{e}} &= \mathbf{Y} - \mathbf{Z}\tilde{\beta} \\
&= \mathbf{Y} - \mathbf{XCC}^{-1}\hat{\beta} \\
&= \mathbf{Y} - \mathbf{X}\hat{\beta} \\
&= \hat{\mathbf{e}}
\end{aligned}$$

where this result holds just by plugging in and canceling terms. This says that the residuals from the regression of $\mathbf{Y}$ on $\mathbf{Z}$ are exactly the same as the residuals from the regression of $\mathbf{Y}$ on $\mathbf{X}$.

As a side-comment, a simple example of this problem would be something like scaling all the regressors by, say, 100. If you did this, it would change the value of the estimated coefficients (divide them by 100) but would fit the data equally well.

Hansen 3.3

$$\begin{aligned}
\mathbf{X}'\hat{\mathbf{e}} &= \mathbf{X}'(\mathbf{Y} - \mathbf{X}\hat{\beta}) \\
&= \mathbf{X}'(\mathbf{Y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}) \\
&= \mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}) \\
&= \mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{Y} \\
&= \mathbf{0}
\end{aligned}$$

which holds by plugging in for $\hat{\mathbf{e}}$ and canceling terms.

Hansen 3.5

The OLS coefficient from a regression of $\hat{\mathbf{e}}$ on $\mathbf{X}$ is given by

$$
\begin{aligned}
(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\hat{\mathbf{e}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{Y}-\mathbf{X}\hat{\beta}) \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\hat{\beta} \\
&= \hat{\beta} - \hat{\beta} \\
&= 0
\end{aligned}
$$

where the first part of the third equality holds by the definition of $\hat{\beta}$ and the last part holds by canceling the terms involving $(\mathbf{X}'\mathbf{X})$.

Hansen 3.7

$$
\begin{aligned}
\mathbf{PX} &= \mathbf{P}\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2\end{bmatrix} \\
&= \begin{bmatrix}\mathbf{PX}_1 & \mathbf{PX}_2\end{bmatrix}
\end{aligned}
$$

Further, since $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2\end{bmatrix}$ and $\mathbf{PX} = \mathbf{X}$ (from the properties of the projection matrix $\mathbf{P}$), this implies that $\mathbf{PX}_1 = \mathbf{X}_1$.

Similarly,

$$
\begin{aligned}
\mathbf{MX} &= \mathbf{M}\begin{bmatrix}\mathbf{X}_1 & \mathbf{X}_2\end{bmatrix} \\
&= \begin{bmatrix}\mathbf{MX}_1 & \mathbf{MX}_2\end{bmatrix}
\end{aligned}
$$

but we also know that $\mathbf{MX} = \mathbf{0}_{n\times k} = \begin{bmatrix}\mathbf{0}_{n\times k_1} & \mathbf{0}_{n\times k_2}\end{bmatrix}$ where, for example, $\mathbf{0}_{n\times k_1}$ is an $n\times k_1$ matrix of zeroes. This implies that $\mathbf{MX}_1 = \mathbf{0}_{n\times k_1}$.