Basic Probability Theory

This material comes from Hansen's *Probability and Statistics for Economists* (PSE) and Len Goff's lecture notes along with some of my own comments.

Introduction

PSE 1.1

Probability is the mathematical language for dealing with **uncertainty** or **random** events (will define these more precisely soon, but it is fine to think about them informally for now). Some things are obviously uncertain and connected with probability; e.g., flipping a coin, rolling dice, card games or other games of chance. Other, more complicated applications that seem obviously connected to probability are things like forecasting; for example, it seems natural to think of tomorrow's weather or the stock price of some company one year in the future as being uncertain/random. We will use these sort examples some during the first few weeks of the course, but these sorts of applications are not what most researchers in economics and other business disciplines (at least those working with data) are most interested in studying.

Instead, in econometrics (and statistics more generally), the most common type of problem that we'll be interested in is that there is some feature of a large population that we'd like to know about, and we want to use data (that is, random draws from the population) to learn something about the feature of the population that we are interested in. A good example of this would be if you were a labor economist interested in studying the unemployment rate in the United States. In principle, there is a "true" unemployment rate in the U.S. But it would be hard to figure out exactly what it is; it would be costly to track down everyone person in the U.S. and figure out their employment status. And, even if you could do it, it would be challenging to do it quickly enough so that some people did not change their employment status since the first time you asked them. Instead, in order to track the unemployment rate, the Census Bureau takes a survey; that is, they (attempt to) randomly select a subset of individuals in the U.S. (I think about 60,000 per month but I'm not 100% sure) their employment status. From this sample, they report an estimate of the unemployment rate along with a measure of **statistical uncertainty** (roughly: how different would it be reasonable to think that their estimate could be from "true" unemployment rate).

Going back to uncertainty, for a particular person, it may not feel like their employment status is random (in particular, a person knows whether or not they are employed). But if you take the view of the researcher, employment status is random. Supposing that we have some way to randomly "draw" a person from the population of people in the U.S., whether this person is employed or not is random to us (sort of like the examples of weather and stock prices above).

We might also be interested in how unemployment is correlated with people's characteristics. Does it vary by age or race or region of the country? These sorts of questions are also related to some ideas in probability.

We will come back to issues related to data in a big way over the course of the next few weeks.

For now, we'll concern ourselves learning some useful tools concerning probability.

Outcomes and Events

PSE 1.2

To start with, let's define some terms:

- An **outcome** is the result of a random process. For example, the outcome of flipping a coin is whether we actually flipped a heads or tails.
- The **sample space** is the set of all possible outcomes; we will denote the sample space by S. For flipping a coin, the sample space is $S = \{H, T\}$ (for heads and tails). For rolling a die, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

  As examples of economic variables, the sample space for a person's yearly income: $S = [0, \infty)$; the sample space for a firm's industry: $S={\textrm{manufacturing}, {service}, other}
- An **event** A is a subset of outcomes in S. For example, the event of rolling an even number on a single dice roll is $A = \{2, 4, 6\}$. The event of rolling a 1 is $A = \{1\}$. Another possible event is $A = S$ (i.e., the whole sample space). Soon we'll define probabilities of events.

Before we get to defining probability itself, following the textbook, let's define a few more things:

Review of Operations on Sets (Definition 1.1)

1. A is a **subset** of B, written $A \subset B$, if every element of A is an element of B. In math: $x \in A \implies x \in B$.
   - Two sets are said to be **equal**, that is $A = B$, if $A \subset B$ and $B \subset A$.
2. The event with no outcomes $\varnothing = \{\}$ is called the **empty set**.
3. The **union** $A \cup B$ is the collection of all outcomes that are in either A **or** B (or both). In match: $A \cup B = \{x : x \in A \text{ or } x \in B\}$. Note: you should read the ":" as "such that".
4. The **intersection** $A \cap B$ is the collection of elementat that are in both A **and** B. In math: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
5. The **complement** A^c of A are all outcomes in S which are not in A. In math: $A^c = \{x : x \notin A\}$.
6. The events A and B are disjoint if they have no outcomes in common; that is, if $A \cap B = \varnothing$
7. The events $A_1, A_2, \ldots$ are a **partition** of S if they are mutually disjoint and their union is S; that is, if $A_i \cap A_j = \varnothing$ for all $i \neq j$ and $\bigcap_{i=1}^{\infty} A_i = S$.

Example: Rolling a die

$$S = \{1, 2, 3, 4, 5, 6\}$$

Examples of events:

$$A = \{1\}, \quad B = \{2\}, \quad C = \{2, 4, 6\}$$

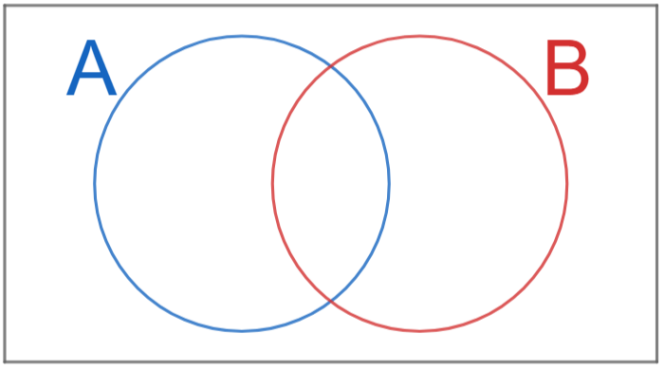
Then,

$$A \cup C = \{1, 2, 4, 6\}$$
$$A \cap C =$$
$$B \cap C = \{2\}$$
$$C^c = \{1, 3, 5\}$$

Properties of Set Operations For any three events A, B, C defined on sample space S, the following properties hold:

- **Commutative:** $A \cup B = B \cup A$ and $A \cap B = B \cap A$
- **Associative:** $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.
- **Distributive:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **DeMorgan's Law:** $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

It's relatively straightforward to write proofs of all of these. We'll write a proof of DeMorgan's Law momentarily. Before we do that, let me mention **Venn Diagrams**. Typically, they look something like this



In my view, writing down a Venn diagram is not a proof, but it can be useful to clarify your thinking when working with sets. For example, it is fairly easy to map DeMorgan's law into the picture above (you can try this as practice).

Now for a proof of DeMorgans law. To start with, let's define two new sets, $C = (A \cup B)^c$ and $D = A^c \cap B^c$; we want to show that $C = D$, and we will show this by showing (i) $C \subset D$ and (ii) $D \subset C$. To show the first part, notice that $x \in C \implies x \notin (A \cup B) \implies x \notin A$ and $x \notin B \implies x \in A^c$ and $x \in B^c \implies x \in (A^c \cap B^c) \implies x \in D$. Thus, $C \subset D$.

For the second part, notice that $x \in D \implies x \in A^c$ and $x \in B^c \implies x \notin A$ and $x \notin B \implies x \notin (A \cup B) \implies x \in (A \cup B)^c \implies x \in D$. Thus, $D \subset C$, and we have completed the proof.

Probability Function

PSE 1.3-1.4

A **probability function** P is a function that that takes in events and returns probabilities (numbers from 0 to 1). As a side-comment, I am going to side-step discussing some technical details related to exactly which events probabilities can be defined for (if you are interested, see the discussion in PSE 1.14 on sigma fields). This will basically amount to ruling out some uncommon cases that can happen when there are an uncountably infinite number of possible events.

Here is a more formal definition. A **probability function** $\mathrm{P}(\cdot)$ is a function from events to $\mathbb{R}$, satisfying the following properties (which are referred to as the **Axioms of Probability**):

1. $\mathrm{P}(A) \geq 0$. In words: the probability of any event (and therefore probabilities in general) is non-negative
2. $\mathrm{P}(S) = 1$. In words: the probability of some outcome in the sample space (recall: this is the set of all possible outcomes) is equal to 1.
3. If $A_1, A_2 \ldots$ is a countable collection of disjoint sets, then

$$\mathrm{P}\left(\bigcup_j A_j\right) = \sum_j \mathrm{P}(A_j)$$

   In words: Probabilities are additive on disjoint events.

Side-comment: The textbook also has an interesting discussion of the meaning of probabilities. To briefly summarize, there are two main ways to ascribe meaning to probabilities. Perhaps the most common one, called *frequentism*, is based on the long run frequencies (e.g., the probability of flipping a heads is 50% because if you did it a tremendously large number of times, the "long run probability" of flipping a heads is 50%). The other main view is called *subjectivism* which basically refers to an individual's "degree of belief" about the uncertainty of an event.

The first two parts of the definition of the probability function seem straightforward. For the third part of the definition, consider rolling a die. Suppose we are interested in the event $A = \{2, 4, 6\}$ (i.e., that we roll an even number). Notice that we can re-write $A = \{2\} \cup \{4\} \cup \{6\}$, and that these

are all disjoint events (this is because if you roll a 2, it means that you could not have rolled a 4). The third part of the definition implies that we can write $\mathrm{P}(A) = \mathrm{P}(\{2\}) + \mathrm{P}(\{4\}) + \mathrm{P}(\{6\}) = 1/6 + 1/6 + 1/6 = 1/2$.

These axioms imply several intuitive properties of probability. As an example, let's start by showing: $\mathrm{P}(A^c) = 1 - \mathrm{P}(A)$. In words, this says that the probability of all outcomes that are not in A is equal to one minus the probability of A.

To show this, notice that A and A^c are disjoint sets. Moreover, $A \cup A^c = S$. Thus, $1 = \mathrm{P}(S) = \mathrm{P}(A) + \mathrm{P}(A^c)$ where the first equality holds by the second property of a probability function and the second equality holds by the third property of a probability function. This implies the result.

Practice: Using the definition of a probability function, show the following

1. Show that if $A \subseteq B$: $\mathrm{P}(A) \leq \mathrm{P}(B)$.
2. Use the above to show that $\mathrm{P}(A \cap B) \leq \min\{\mathrm{P}(A), \mathrm{P}(B)\}$.
3. Derive the expression: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.
4. Derive the expression: $\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)$. *Hint:* use $(A \cap B)^c = A^c \cup B^c$.

See also Theorem 1.2 in the textbook (as well as Section 1.15) for more examples.

Joint events, conditional probability, and independence

PSE 1.6-1.8

At the beginning of these notes, we briefly mentioned that one might be interested in how events are related to each other. One of the examples in the textbook is where A is the event of "making an A on an econometrics exam" and B is the even of "studying 12 hours a day", and where a natural question is how event B affects the probability of event A. Another example is the one from earlier in the notes where one event might be "being unemployed" and another event might be "being younger than 35". To think along these lines, we'll define conditional probabilities; that is, the probability that event A occurs conditional on event B occurring.

Definition:
iven an event B such that $\mathrm{P}(B) > 0$, the **conditional probability** of event A given B is defined as

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

A natural question at this point is why we define a conditional probability this way. Recall that the intersection of two events $A \cap B$ can be interpreted as the event that *both* of events A and B occur — this corresponds to the term in the numerator. And you might be tempted to think that

this is the conditional probability. The textbook refers to dividing by $\mathrm{P}(B)$ as a normalization to the "new" sample space B; as an additional comment, it seems helpful to think about a relatively extreme example. Consider the example of studying for an econometrics exam above. Suppose that all students that study for 12 hours a day make an A (this immediately suggests that the conditional probability of making an A conditional on 12 hours per day studying should be equal to 1). However, suppose that only 5% of students study for 12 hours a day. In this case, $\mathrm{P}(A \cap B) = \mathrm{P}(B) = 0.05$ (which is not equal to 1). However, $\mathrm{P}(A \cap B)/\mathrm{P}(B) = \mathrm{P}(B)/\mathrm{P}(B) = 1$ which is as it should be.

Another very important concept in probability is when events are **independent**. Intuitively, events are independent if knowing the outcome of one event does not provide any information about the probability that another event occurred. In math, we say that two events A and B are independent if

$$\mathrm{P}(A|B) = \mathrm{P}(A)$$

or, in other words, the conditional probability of A conditional on B is equal to the unconditional probability of A.

An implication of two events being independent (often, this is actually given as the definition of events being independent though I think the previous one is a bit more straightforward to understand) is that

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$$

To see this, notice that

$$\begin{aligned}\mathrm{P}(A \cap B) &= \mathrm{P}(A|B)\mathrm{P}(B) \\ &= \mathrm{P}(A)\mathrm{P}(B)\end{aligned}$$

where the first line holds by rearranging the definition of conditional probability (as a side-comment, this is a useful/common step so it's one to pay attention to) and the second equality holds because A and B are independent.

It's natural to think about things like outcomes of two flips of a coin or rolls of dice as being independent though most of the examples that you'd find interesting in economics/business/etc. are unlikely to be independent. Section 1.8 in the textbook walks through some examples that you might find helpful.

Law of total probability and Bayes' rule

PSE 1.9-1.10

Another useful result for working with probabilities and multiple events is called the **law of total probability**: If $B_1, B_2, \ldots$ is a partition of the sample space and $\mathrm{P}(B_i) > 0$ for all i, then for any event A: $\mathrm{P}(A) = \sum_i \mathrm{P}(A|B_i)\mathrm{P}(B_i)$.

Proof:

$$\begin{aligned}
\mathrm{P}(A) &= \mathrm{P}(A \cap S) \\
&= \mathrm{P}\left(A \cap (\cup_i B_i)\right) \\
&= \mathrm{P}\left(\cup_i (A \cap B_i)\right) \\
&= \sum_i \mathrm{P}(A \cap B_i) \\
&= \sum_i \mathrm{P}(A|B_i)\mathrm{P}(B_i)
\end{aligned}$$

where the first line holds since any event $A \subseteq S$, so that $A = A \cap S$ and thus $\mathrm{P}(A) = \mathrm{P}(A \cap S)$. The second line holds because $B_1, B_2, \ldots$ forms a partition of S. The third equality holds because $A \cap (\bigcup_i B_j) = \bigcup_i (A \cap B_i)$. The fourth line holds because the events $(A \cap B_i)$ are disjoint for different values of i (since each is a subset of B_i) and by applying the third part of the definition of probability. And the last equality holds by the definition of conditional probability.

Finally, we will cover a famous result in probability called **Bayes' rule**. Bayes' rule says that, if $\mathrm{P}(A) > 0$ and $\mathrm{P}(B) > 0$), then

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B|A)\mathrm{P}(A) + \mathrm{P}(B|A^c)P(A^c)}$$

Proof:

The proof essentially follow from repeatedly applying the definition of conditional probability and from the law of total probability.

$$\begin{aligned}
\mathrm{P}(A|B) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \\
&= \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B)} \\
&= \frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B|A)P(A) + \mathrm{P}(B|A^c)\mathrm{P}(A^c)}
\end{aligned}$$

where the first equality uses the definition of conditional probability, the second equality uses the (rearranged version of) the definition of conditional probability, and the third equality uses the law of total probability (notice that A and A^c form a partition of the sample space).