

## Homework 3 Solutions

### PSE 4.11

To calculate the correlation, we'll calculate all of the components of

$$\text{Corr}(XY, Y) = \frac{\text{Cov}(XY, Y)}{\sqrt{\text{Var}(XY)\text{Var}(Y)}}$$

Notice that

$$\begin{aligned}\text{cov}(XY, Y) &= \mathbb{E}[XY^2] - \mathbb{E}[XY]\mathbb{E}[Y] \\ &= \mathbb{E}[X]\mathbb{E}[Y^2] - \mathbb{E}[X]\mathbb{E}[Y]^2 \\ &= \mathbb{E}[X](\mathbb{E}[Y^2] - \mathbb{E}[Y]^2) \\ &= \mu_X \sigma_Y^2\end{aligned}$$

where the second equality uses independence. Next,

$$\begin{aligned}\text{var}(XY) &= \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2 \\ &= \mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}[X]^2\mathbb{E}[Y]^2 \\ &= (\mathbb{E}[X^2] - \mathbb{E}[X]^2)\mathbb{E}[Y^2] + \mathbb{E}[X]^2(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2) \\ &= \sigma_X^2\mathbb{E}[Y^2] + \mu_X^2\sigma_Y^2 \\ &= \sigma_X^2(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2) + \sigma_X^2\mathbb{E}[Y]^2 + \mu_X^2\sigma_Y^2 \\ &= \sigma_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \mu_X^2\sigma_Y^2\end{aligned}$$

where the second equality holds by independence, the third equality by adding and subtracting  $\mathbb{E}[X]^2\mathbb{E}[Y^2]$ , the fourth equality by the definitions of  $\sigma_X^2$ ,  $\sigma_Y^2$ , and  $\mu_X$ , the fifth equality by adding and subtracting  $\sigma_X^2\mathbb{E}[Y]^2$ , and the last equality by the definitions of  $\sigma_Y^2$  and  $\mu_Y$ . Finally,

$$\text{Var}(Y) = \sigma_Y^2$$

Putting these expressions into the original expression for  $\text{Corr}(XY, Y)$ , we have that

$$\begin{aligned}\text{Corr}(X, Y) &= \frac{\mu_X \sigma_Y^2}{\sqrt{\sigma_Y^2} \sqrt{\sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \mu_X^2}} \\ &= \frac{\mu_X \sigma_Y}{\sqrt{\sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \mu_X^2}}\end{aligned}$$

### PSE 4.15

a) To find  $\mathbb{E}[Y]$ , we can use the law of iterated expectations

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[X] = \frac{1}{2}$$

where the first equality holds by the law of iterated expectations, the second equality because  $\mathbb{E}[Y \mid X = x] = x$  and the last equality because  $X$  is  $U[0, 1]$ .

b) To find  $\text{Var}(Y)$ , we can use the law of total variance

$$\text{Var}(Y) = \mathbb{E}[\text{Var}(Y | X)] + \text{Var}(\mathbb{E}[Y | X])$$

We can consider each of these in turn. Since  $\text{Var}(Y|X = x) = x^2$ , the first term  $\mathbb{E}[\text{Var}(Y | X)] = \mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = 1/12 + (1/2)^2 = 4/12 = 1/3$ .

For the second term,  $\text{Var}(\mathbb{E}[Y | X]) = \text{Var}(X) = 1/12$ .

Adding these together, we have that  $\text{Var}(Y) = 1/3 + 1/12 = 5/12$ .

## PSE 5.2

The pdf of a standard normal random variable is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Thus,

$$\phi'(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) (-x) = -x\phi(x)$$

where the first equality holds by the chain rule, and the second equality uses the definition of  $\phi(x)$ .

## PSE 5.5

$$\begin{aligned} M_Z(t) &= \mathbb{E}[\exp(tZ)] \\ &= \int_{-\infty}^{\infty} \exp(tz) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2 - 2tz + t^2 - t^2}{2}\right) dz \\ &= \exp\left(\frac{t^2}{2}\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-t)^2}{2}\right) dz}_{=1} \\ &= \exp\left(\frac{t^2}{2}\right) \end{aligned}$$

where the first equality holds by the definition of moment generating function, the second equality writes the expectation as an integral and uses the pdf of  $Z \sim \mathcal{N}(0, 1)$ , the third equality combines the two exponential terms and then adds and subtracts  $t^2/2$  inside the exponential, the fourth equality moves the part of the exponential that does not depend on  $z$  out of the integral and then uses that  $z^2 - 2tz + t^2 = (z - t)^2$ , and the last equality holds by noticing that the underlined term equals one because it integrates over a pdf (in particular, the pdf of a  $\mathcal{N}(t, 1)$  random variable). The expression in the last line is the answer to the problem.

## PSE 6.2

Consider the following estimator of  $\mu'_k = \mathbb{E}[X^k]$ :

$$\hat{\mu}'_k = \frac{1}{n} \sum_{i=1}^n X_i^k$$

Notice that

$$\begin{aligned} \mathbb{E}[\hat{\mu}'_k] &= \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n X_i^k \right] \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i^k] \\ &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X^k] \\ &= \frac{1}{n} \sum_{i=1}^n \mu'_k \\ &= \mu'_k \end{aligned}$$

where the first equality holds by the definition of  $\hat{\mu}'_k$ , the second equality holds because constants can come out of expectations and because expectations can pass through sums, the third equality holds because the  $X_i$  are identically distributed, the fourth equality holds by the definition of  $\mu'_k$ , and the last equality holds because  $\mu'_k$  is constant. This shows that  $\hat{\mu}'_k$  is unbiased for  $\mu'_k$ .

#### PSE 6.4

$$\begin{aligned} \text{Var}(\hat{\mu}'_k) &= \text{Var} \left( \frac{1}{n} \sum_{i=1}^n X_i^k \right) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i^k) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X^k) \\ &= \frac{\text{Var}(X^k)}{n} \end{aligned}$$

where the first equality holds by the definition of  $\hat{\mu}'_k$ , the second equality holds because the  $X_i$  are independent (hence no covariance terms), the third equality holds because the  $X_i$  are identically distributed, and the last equality simplifies the previous expression.

#### PSE 6.9

a)  $\hat{\theta} = \exp(\bar{X})$

b)  $\hat{\theta} = \log \left( \frac{1}{n} \sum_{i=1}^n \exp(X_i) \right)$

c)  $\hat{\theta} = \sqrt{\frac{1}{n} \sum_{i=1}^n X_i^4}$

d) It's helpful to re-write  $\text{var}(X^2) = \mathbb{E}[(X^2)^2] - \mathbb{E}[X^2]^2$  which suggests the following estimator

$$\widehat{\text{var}}(X^2) = \frac{1}{n} \sum_{i=1}^n X_i^4 - \left( \frac{1}{n} \sum_{i=1}^n X_i^2 \right)^2$$