Homework 2 Solutions

PSE 1.1

a) $A \cap B = \{a, c\}$

b) $A \cup B = \{a, b, c, d, e, f\}$

PSE 1.2

a) $S = \{H, T\}$

b) $S = \{1, 2, 3, 4, 5, 6\}$

c) $S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$ (where the first digit indicates the outcome on the first roll and the second digit indicates the outcome of the second roll)

d) There are a lot of possibilities here, so I decided to write a small piece of code to write down all the elements of the sample space

```r
# function to add next free throw (either make: I or miss: O)
next_freethrow <- function(previous_freethrows) {

  # stop if have already shot 6 free throws
  if (stringr::str_length(previous_freethrows[1]) == 6) {
    return(previous_freethrows)
  }

  # add make and miss to previous list
  next_added <- unlist(lapply(previous_freethrows, function(pf) {
    c(paste0(pf,"I"), paste0(pf,"O"))
  }))

  # recursively call function on updated sample space of free throws
  next_freethrow(next_added)
}

# to generate sample space, call: nextfreethrow("")
```

The sample space is $S = \{$ IIIIII, IIIIIO, IIIIOI, IIIIOO, IIIOII, IIIOIO, IIIOOI, IIIOOO, IIOIII, IIOIIO, IIOIOI, IIOIOO, IIOOII, IIOOIO, IIOOOI, IIOOOO, IOIIII, IOIIIO, IOIIOI, IOIIOO, IOIOII, IOIOIO, IOIOOI, IOIOOO, IOOIII, IOOIIO, IOOIOI, IOOIOO, IOOOII, IOOOIO, IOOOOI, IOOOOO, OIIIII, OIIIIO, OIIIOI, OIIIOO, OIIOII, OIIOIO, OIIOOI, OIIOOO, OIOIII, OIOIIO, OIOIOI, OIOIOO, OIOOII, OIOOIO, OIOOOI, OIOOOO, OOIIII, OOIIIO, OOIIOI, OOIIOO, OOIOII, OOIOIO, OOIOOI, OOIOOO, OOOIII, OOOIIO, OOOIOI, OOOIOO, OOOOII, OOOOIO, OOOOOI, OOOOOO $\}$ (where I stands for "In" (i.e., made the shot) and O stands for "Out" (i.e., missed the shot)). There are 64 ($= 2^6$) elements in the sample space.

PSE 1.5

No, for A and B to be disjoint, it must be that $\mathrm{P}(A \cap B) = 0$. From the result of PSE 1.6 below, we have that

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

If A and B are disjoint, then the RHS of the previous equation is equal to: $\mathrm{P}(A \cup B) = 1/2 + 2/3 - 0 = 7/6 > 1$. But probabilities can't be greater than 1, therefore, A and B cannot be disjoint.

PSE 1.6

First, using the same arguments as for the partitioning theorem, $x \in A \implies x \in A \cap B$ or $x \in A \cap B^C$ and that these are disjoint sets. Therefore, by the third axiom of probability, $\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^C)$. Exactly the same sort of argument implies that $\mathrm{P}(B) = \mathrm{P}(A \cap B) + \mathrm{P}(A^C \cap B)$.

Next, let $C = A \cup B$. Then, we have that

$$\begin{aligned} x \in C &\implies x \in A \text{ or } x \in B \\ &\implies (x \in A \cap B \text{ or } x \in A \cap B^C) \text{ or } (x \in A \cap B \text{ or } x \in A^C \cap B) \\ &\implies x \in A \cap B \text{ or } x \in A \cap B^C \text{ or } x \in A^C \cap B \end{aligned}$$

where the second line holds from the partitioning theorem and where the sets in the last line are disjoint sets. Therefore, by the third axiom of probability

$$\mathrm{P}(C) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^C) + \mathrm{P}(A^C \cap B)$$

Then, notice that by combining the previous terms, we have that

$$\begin{aligned} \mathrm{P}(C) - \mathrm{P}(A) - \mathrm{P}(B) &= \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^C) + \mathrm{P}(A^C \cap B) \\ &\qquad - \Big(\mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^C) + \mathrm{P}(A \cap B) + \mathrm{P}(A^C \cap B)\Big) \\ &= -\mathrm{P}(A \cap B) \end{aligned}$$

Thus, $\mathrm{P}(C) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ which is the desired result.

PSE 1.17

Let B denote the event that an athlete takes a banned substance and let A denote the event that an athlete tests positive. We are interested in $\mathrm{P}(B|A)$. From the problem, we know that $P(B) = 0.01$, $P(A|B) = 0.4$ and that $\mathrm{P}(A^C|B) = 0.01$.

From the definition of conditional probability, we know that

$$\begin{aligned} \mathrm{P}(B|A) &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} \\ &= \frac{\mathrm{P}(A|B)\mathrm{P}(B)}{\mathrm{P}(A|B)\mathrm{P}(B) + \mathrm{P}(A|B^C)\mathrm{P}(B^C)} \end{aligned}$$

(which amounts to just re-deriving Bayes rule), and we know each term above; in particular,

$$\mathrm{P}(B|A) = \frac{(0.4)(0.01)}{(0.4)(0.01) + (0.01)(0.99)} \approx 0.29$$

In other words, the probability that the athlete took the banned substance conditional on testing positive is only about 29%.

PSE 1.18

To show that A and B are independent, notice that $\mathrm{P}(A) = \mathrm{P}(B) = 1/6$; thus $\mathrm{P}(A) \cdot \mathrm{P}(B) = 1/36$. Now consider $\mathrm{P}(A \cap B) = \mathrm{P}(\{66\})$ (i.e., the probability of rolling a 6 and then a second 6). There are 36 equally likely outcomes of rolling two dice; therefore $\mathrm{P}(A \cap B) = 1/36$. Thus A and B are (unconditionally) independent.

Next, consider $\mathrm{P}(A|C) = \mathrm{P}(A \cap C)/\mathrm{P}(C)$. Moreover, $\mathrm{P}(A \cap C) = \mathrm{P}(\{66\}) = 1/36$ and $\mathrm{P}(C) = \mathrm{P}(\{11, 22, 33, 44, 55, 66\}) = 1/6$; thus $\mathrm{P}(A|C) = 1/6$. A similar argument implies that $\mathrm{P}(B|C) = 1/6$. Therefore, $\mathrm{P}(A|C) \cdot \mathrm{P}(B|C) = 1/36$.

Finally, $\mathrm{P}(A \cap B|C) = \mathrm{P}(A \cap B \cap C)/\mathrm{P}(C)$. $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(\{66\}) = 1/36$. Recalling that $\mathrm{P}(C) = 1/6$, we have that $\mathrm{P}(A \cap B|C) = 1/6 \neq 1/36 = \mathrm{P}(A|C) \cdot \mathrm{P}(B|C)$; thus, A and B are not independent conditional on C.

2.3

a) To show that $\mathrm{F}(x)$ is a valid cdf, we verify the four properties of a cdf. First, we show that the cdf is non-decreasing. Notice that $f(x) = \frac{d}{dx}\mathrm{F}(x) = \exp(-x)$, which is greater than 0 for all $x \geq 0$. That its derivative is non-negative implies that the cdf is non-decreasing.

   Second, since $\mathrm{F}(x)$ is defined to be 0 for all $x < 0$, we have that $\lim_{x \to -\infty} \mathrm{F}(x) = 0$, which is the second property of a cdf. Third, $\lim_{x\to\infty} \mathrm{F}(x) = \lim_{x\to\infty} 1 - \exp(-x) = 1$, which is the third property of a cdf.

   Finally, $\mathrm{F}(x)$ is right-continuous since it is continuous for all x.

b) We found the pdf of X in part (a), given by $f(x) = \exp(-x)$ for $x \geq 0$.

c) The expected value of X is given by

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty x \exp(-x)\, dx \\
&= \Big[-x\exp(-x)\Big]_0^\infty + \int_0^\infty \exp(-x)\, dx \\
&= \lim_{t\to\infty}\left(\frac{1}{\exp(t)}\right) - 0 + [-\exp(-x)]_0^\infty \\
&= 0 + 1 = 1
\end{aligned}$$

   where the first equality holds by writing the expectation as an integral, the second equality holds by integration by parts, i.e., $\int_0^\infty u\, dv = [uv]_0^\infty - \int_0^\infty v\, du$ with $u = x$ and $dv = \exp(-x)\, dx \implies du = dx$ and $v = -\exp(-x)$, the third equality holds by evaluating the limits for the first term (the limit part comes from L'Hopital's rule) and the second term by direct evaluation, and the fourth line holds by direct calculation.

2.6

a) I think there is a mistake in the pdf given in part (a) and that the pdf should be $f(x) = ax^{a-1}$ for $0 < x < 1$ and 0 otherwise, with $a > 0$ (i.e., I think there is a typo where the $-a$ in the exponent should just be a). I'll solve the problem for this corrected pdf.

For the expectation, we have that

$$\begin{aligned}\mathbb{E}[X] &= \int_0^1 x \cdot ax^{a-1}\, dx \\ &= a \int_0^1 x^a\, dx \\ &= a \left[\frac{x^{a+1}}{a+1}\right]_0^1 = a/(a+1)\end{aligned}$$

For the variance, first notice that

$$\begin{aligned}\mathbb{E}[X^2] &= \int_0^1 x^2 \cdot ax^{a-1}\, dx \\ &= a \int_0^1 x^{a+1}\, dx \\ &= a \left[\frac{x^{a+2}}{a+2}\right]_0^1 = a/(a+2)\end{aligned}$$

and, thus, we have that

$$\begin{aligned}\mathrm{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= a/(a+2) - (a/(a+1))^2\end{aligned}$$

b) Starting with the expectation, we have that

$$\mathbb{E}[X] = \sum_{x=1}^{n} x \cdot \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x = \frac{1}{n} \cdot \frac{n(n+1)}{2} = (n+1)/2$$

where the third equality uses the trick that $\sum_{x=1}^{n} x = 1+2+...+n$, and also $\sum_{x=1}^{n} x = n+(n-1)+...+1$, and adding these two sums together gives $2\sum_{x=1}^{n} x = (n+1)+(n+1)+...+(n+1) = n(n+1)$, and dividing both sides by 2 gives the desired result.

Next, for the variance, notice that

$$\mathbb{E}[X^2] = \sum_{x=1}^{n} x^2 \cdot \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} = (n+1)(2n+1)/6$$

where the third equality uses a similar (though more complicated) trick as for the expectation above.

Then, we have that

$$
\begin{aligned}
\operatorname{var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= (n+1)(2n+1)/6 - ((n+1)/2)^2 \\
&= (n+1)(2n+1)/6 - (n+1)^2/4 \\
&= (n+1)\left((2n+1)/6 - (n+1)/4\right) \\
&= (n+1)\left((4(2n+1) - 6(n+1))/24\right) \\
&= (n+1)\left((8n+4-6n-6)/24\right) \\
&= (n+1)\left((2n-2)/24\right) = (n^2-1)/12
\end{aligned}
$$