Homework 2 Solutions

PSE 1.1

a) $A \cap B = \{a, c\}$

b) $A \cup B = \{a, b, c, d, e, f\}$

PSE 1.2

a) $S = \{H, T\}$

b) $S = \{1, 2, 3, 4, 5, 6\}$

c) $S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$ (where the first digit indicates the outcome on the first roll and the second digit indicates the outcome of the second roll)

d) There are a lot of possibilities here, so I decided to write a small piece of code to write down all the elements of the sample space

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# function to add next free throw (either make: I or miss: O)
next_freethrow <- function(previous_freethrows) {

  # stop if have already shot 6 free throws
  if (stringr::str_length(previous_freethrows[1]) == 6) {
    return(previous_freethrows)
  }

  # add make and miss to previous list
  next_added <- unlist(lapply(previous_freethrows, function(pf) {
    c(paste0(pf,"I"), paste0(pf,"O"))
  }))

  # recursively call function on updated sample space of free throws
  next_freethrow(next_added)
}

# to generate sample space, call: nextfreethrow("")
```

The sample space is $S = \{$ IIIIII, IIIIIO, IIIIOI, IIIIOO, IIIOII, IIIOIO, IIIOOI, IIIOOO, IIOIII, IIOIIO, IIOIOI, IIOIOO, IIOOII, IIOOIO, IIOOOI, IIOOOO, IOIIII, IOIIIO, IOIIOI, IOIIOO, IOIOII, IOIOIO, IOIOOI, IOIOOO, IOOIII, IOOIIO, IOOIOI, IOOIOO, IOOOII, IOOOIO, IOOOOI, IOOOOO, OIIIII, OIIIIO, OIIIOI, OIIIOO, OIIOII, OIIOIO, OIIOOI, OIIOOO, OIOIII, OIOIIO, OIOIOI, OIOIOO, OIOOII, OIOOIO, OIOOOI, OIOOOO, OOIIII, OOIIIO, OOIIOI, OOIIOO, OOIOII, OOIOIO, OOIOOI, OOIOOO, OOOIII, OOOIIO, OOOIOI, OOOIOO, OOOOII, OOOOIO, OOOOOI, OOOOOO $\}$ (where I stands for "In" (i.e., made the shot) and O stands for "Out" (i.e., missed the shot)). There are 64 ($= 2^6$) elements in the sample space.

PSE 1.5

No, for A and B to be disjoint, it must be that $\mathrm{P}(A \cap B) = 0$. From the result of PSE 1.6 below, we have that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are disjoint, then the RHS of the previous equation is equal to: $P(A \cup B) = 1/2 + 2/3 - 0 = 7/6 > 1$. But probabilities can't be greater than 1, therefore, A and B cannot be disjoint.

PSE 1.6

First, using the same arguments as for the partitioning theorem, $x \in A \implies x \in A \cap B$ or $x \in A \cap B^C$ and that these are disjoint sets. Therefore, by the third axiom of probability, $P(A) = P(A \cap B) + P(A \cap B^C)$. Exactly the same sort of argument implies that $P(B) = P(A \cap B) + P(A^C \cap B)$.

Next, let $C = A \cup B$. Then, we have that

$$\begin{aligned} x \in C &\implies x \in A \text{ or } x \in B \\ &\implies (x \in A \cap B \text{ or } x \in A \cap B^C) \text{ or } (x \in A \cap B \text{ or } x \in A^C \cap B) \\ &\implies x \in A \cap B \text{ or } x \in A \cap B^C \text{ or } x \in A^C \cap B \end{aligned}$$

where the second line holds from the partitioning theorem and where the sets in the last line are disjoint sets. Therefore, by the third axiom of probability

$$P(C) = P(A \cap B) + P(A \cap B^C) + P(A^C \cap B)$$

Then, notice that by combining the previous terms, we have that

$$\begin{aligned} P(C) - P(A) - P(B) &= P(A \cap B) + P(A \cap B^C) + P(A^C \cap B) \\ &\quad - \Big(P(A \cap B) + P(A \cap B^C) + P(A \cap B) + P(A^C \cap B)\Big) \\ &= -P(A \cap B) \end{aligned}$$

Thus, $P(C) = P(A) + P(B) - P(A \cap B)$ which is the desired result.

PSE 1.17

Let B denote the event that an athlete takes a banned substance and let A denote the event that an athlete tests positive. We are interested in $P(B|A)$. From the problem, we know that $P(B) = 0.01$, $P(A|B) = 0.4$ and that $P(A^C|B) = 0.01$.

From the definition of conditional probability, we know that

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^C)P(B^C)} \end{aligned}$$

(which amounts to just re-deriving Bayes rule), and we know each term above; in particular,

$$P(B|A) = \frac{(0.4)(0.01)}{(0.4)(0.01) + (0.01)(0.99)} \approx 0.29$$

In other words, the probability that the athlete took the banned substance conditional on testing positive is only about 29%.

PSE 1.18

To show that A and B are independent, notice that $\mathrm{P}(A) = \mathrm{P}(B) = 1/6$; thus $\mathrm{P}(A) \cdot \mathrm{P}(B) = 1/36$. Now consider $\mathrm{P}(A \cap B) = \mathrm{P}(\{66\})$ (i.e., the probability of rolling a 6 and then a second 6). There are 36 equally likely outcomes of rolling two dice; therefore $\mathrm{P}(A \cap B) = 1/36$. Thus A and B are (unconditionally) independent.

Next, consider $\mathrm{P}(A|C) = \mathrm{P}(A \cap C)/\mathrm{P}(C)$. Moreover, $\mathrm{P}(A \cap C) = \mathrm{P}(\{66\}) = 1/36$ and $\mathrm{P}(C) = \mathrm{P}(\{11, 22, 33, 44, 55, 66\}) = 1/6$; thus $\mathrm{P}(A|C) = 1/6$. A similar argument implies that $\mathrm{P}(B|C) = 1/6$. Therefore, $\mathrm{P}(A|C) \cdot \mathrm{P}(B|C) = 1/36$.

Finally, $\mathrm{P}(A \cap B|C) = \mathrm{P}(A \cap B \cap C)/\mathrm{P}(C)$. $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(\{66\}) = 1/36$. Recalling that $\mathrm{P}(C) = 1/6$, we have that $\mathrm{P}(A \cap B|C) = 1/6 \neq 1/36 = \mathrm{P}(A|C) \cdot \mathrm{P}(B|C)$; thus, A and B are not independent conditional on C.