Parametric Distributions

PSE 3.1

In some cases, a researcher may know (or be willing to assume that they know) the distribution of a random variable. More commonly, a researcher might be willing to assume that they know the distribution of a random variable up to a finite number of **parameters**, where the parameters allow for some flexibility in the shape of the distribution. In this section, we will briefly cover some of the most common/useful parametric distributions (though there are a number of other distributions mentioned in this chapter in the textbook that are worth mentioning).

Bernoulli Distribution

PSE 3.2

A random variable that follows a **Bernoulli distribution** takes either the value 0 or 1 with some probability p. An example is whether or not a particular person is employed. In particular, we can write

$$\begin{aligned} \mathrm{P}(X = 1) &= p \\ \mathrm{P}(X = 0) &= 1 - p \end{aligned}$$

which fully summarizes the distribution of X. In practice, in some cases p might be known (e.g., for flipping a coin, $p = 0.5$), but in most interesting cases you would need to somehow estimate p (we'll return to this sort of issue later). It is also useful to have a single, full expression for the pmf of a Bernoulli random variable, which is given by

$$\mathrm{P}(X = x) = p^x (1 - p)^{(1-x)}$$

In particular, just try plugging in $x = 1$ and $x = 0$ here and you will get $\mathrm{P}(X = 1) = p$ and $\mathrm{P}(X = 0) = (1 - p)$ as above.

Two interesting properties of Bernoulli random variables are:

1. $\mathbb{E}[X] = p$, in words: the expected value of X equals p
2. $\mathrm{var}(X) = p(1 - p)$ in words: the variance of X equals $p(1 - p)$.

Here is a proof of the first property, and I'll leave proving the second property as an excerise.

$$\begin{aligned} \mathbb{E}[X] &= \sum_{x \in \{0,1\}} x \mathrm{P}(X = x) \\ &= 0(1 - p) + 1p \\ &= p \end{aligned}$$

Normal Distribution

PSE 3.12

Arguably the most important parametric distribution is the normal distribution. We will soon see that random variables following a standard normal distribution show up naturally in statistics due to the central limit theorem.

A random variable X that follows a normal distribution is continuous and the normal distribution is indexed by two parameters μ (its mean) and σ^2 (its variance). It is common to write $X \sim N(\mu, \sigma^2)$ for a random variable that follow a normal distribution. The pdf of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

A special case (but important case) is when $\mu = 0$ and $\sigma^2 = 1$. In this case, X is said to follow a **standard normal distribution**.

Another useful property of normal distributions is stated in the following propsition

Proposition: Suppose that $X \sim N(\mu, \sigma^2)$ and we define $Y = a + bX$ for some $a, b \in \mathbb{R}$. Then Y is also normally distributed as $N(a + b\mu, b^2\sigma^2)$.

A useful consequence of the previous proposition is that, when $X \sim N(\mu, \sigma^2)$, it can be "standardized" by considering the transformed random variable $Z = (X - \mu)/\sigma$. This sort of transformation will be useful to us in the statistics portion of the course.

There is also some specialized notation that is worth mentioning for standard normal random variables. The cdf of a standard normal distribution is often denoted by Φ, and the pdf is often denoted by ϕ.

Next, if $X \sim N(\mu, \sigma^2)$, then its moment generating function is given by $M_X(t) = \exp(\mu t + \sigma^2 t^2/2)$. For a standard normal random variable, $Z \sim N(0, 1)$, its moment generating function is given by $M_Z(t) = \exp(t^2/2)$.

Normal distributions arise frequently in the context of statistical inference. From your undergraduate statistics class, you may be familiar with the idea of comparing some test statistic to a critical value that comes from a normal distribution. The next table comes from Table 5.1 in the textbook and provides the a summary of the cdf of a standard normal distribution (you'll notice that some of the values of "x" in the table correspond to critical values that you may be familiar with).

Table 5.1: Normal Probabilities and Quantiles

	$\mathbb{P}[Z \leq x]$	$\mathbb{P}[Z > x]$	$\mathbb{P}[\lvert Z \rvert > x]$
$x = 0.00$	0.50	0.50	1.00
$x = 1.00$	0.84	0.16	0.32
$x = 1.65$	0.950	0.050	0.100
$x = 1.96$	0.975	0.025	0.050
$x = 2.00$	0.977	0.023	0.046
$x = 2.33$	0.990	0.010	0.020
$x = 2.58$	0.995	0.005	0.010

In R, to get values of cdfs of standard normal random variable, you can use the `pnorm` function. In R, pre-pending "p" to the name of a distribution recovers values of the cdf for a large number of distributions (e.g., `pchisq` recovers cdfs for chi-square random variables). Similarly, `qnorm` recovers quantiles for a normally distributed random variable (quantiles are the inverse of the cdf; e.g., `qnorm(.7)` would provide the 70th percentile of a standard normal random variable); `rnorm` can be used to make draws from a normally distributed random variable.